Chapter 3
Word Problems

Now that we have an idea on how to solve equations let’s look at a place where will use the idea of equations and how to solve them – word problems.

Need to be able to change word phrases to algebraic expressions. We also need to be able to express one unknown in terms of a second unknown.

A number increased by 24 to results in 36.2 What was the original number?

You can probably find the answer without any algebra – but you need to be able to set up in an algebraic form (equation with variables) so that when you can not see the answer – you can create an equation that is probably solvable.

Change the following phrases to algebraic expressions.

1) Six more than a certain number: ____________________

2) Twelve less than the product of x and y. _________________

3) Twice the sum of x and y. _________________

4) Find the perimeter of a rectangle whose length is 4 more than its width. ________________

5) The sum of two numbers is 24. Write an expression for both numbers. ________________

6) half the product of 4 and x. _________________

7) 24 % of a number. ________________

8) the value in cents of d dimes. ________________

9) the number of months in y years. ________________

10) the number of weeks in d days. ________________
Examples:
Write in terms of x

1. A certain number is six more than a second number. Write an expression for each number.

2. The length of a rectangle is eight meters more than twice its width. Write an expression for the length and width.

3. One number is half the second and that same number (1st) is three times as large as the third number. Write an expression for the three numbers.

4. You invest $4000 into stocks and mutual funds. The amount invested in mutual funds is $400 more than twice the amount invested in stocks. Write an expression for each amount invested.

5. Two numbers differ by 24. Write an expression for each.

6. The cost of a house today is three times its cost in 1970 and it is twice its cost in 1976. Write expressions for each cost in 1970, 1976, and now.

The key to solving problems is to be able to translate word phrases into algebraic expressions and connect the expressions to create algebraic equations.

Read and re-read. Make sure to answer what is being asked instead of just solving the equation you created.

Define your variables
and when necessary draw pictures – tables – whatever helps you to create an equation.
Your author separates Word problems into basic word problems (including those your author classifies as number relation type) - distance-rate-time - and mixture problems.

**Basic Word Problem.**

1. (4/113)  
   A 7:00 A.M. math class has eight fewer students than a 9:00 A.M. class. Find the number of students in each class if the total number of students in both classes is 84.

2. (6/113)  
   The length of a rectangle is five meters more than the width. Find the length and width if the perimeter is 46 meters.

3. (12/114)  
   One number is four more than a second number and this same number is nine less than a third number. Find the three numbers if their sum is 50.

4. (18/115)  
   The price of a certain model of a Texas Instrument calculator is six dollars more than a Sharp calculator and three dollars less than one made by Hewlett-Packard. If the total price of the three calculators is $46.50, find the price of each.

5. (7/121)  
   A student purchases a notebook and a calculator for $14.45. Another student purchases two notebooks and a calculator for $18.95. Find the price of a calculator and the price of a notebook.
6. (8/121)
   A certain number is 12 more than a second number and 20 less than a third number. Find the numbers if their sum is 152.

7. (9/122)
   Three consecutive even integers have a sum of 90. Find the integers.

8. (12/122)
   Alice is 15 centimeters taller than Jane and Alice happens to be 3 centimeters shorter than Diane. The sum of their heights is 489 centimeters. How tall is each?

9. (18/123)
   An individual makes two investments that total $10,000. One investment is at 11 % and the other at 13 %. If the total interest is $1190, find the amount invested at each rate.

10. (11/122)
    Town A has a population ten percent greater than town B. The total population of the two towns is 49,245. Find the population of each.

11. (17/123)
    In a given rectangle the length is twice the width. If the length is increased by seven and the width is increased by eight, the perimeter is doubled. Find the dimensions of the original rectangle.

12. (19/123)
    The base of a given triangle is ten. If the length of the altitude is increased so that it is four more than twice the original, and the base is decreased by two, then the area of the new triangle is twice the area of the original. Find the altitude of the original triangle.
Distance-Rate-Time

Given a constant rate $r$ and a time $t$ we can find what distance an object travels during that time period at the given rate by the formula

$$d = rt$$

units should match accordingly

Examples:

1. How long will it take for person to walk 12 miles if he/she is walking at 3 mph?

2. What distance will a car travel in 45 minutes if it is traveling at 64 mph?

3. At what rate will a bicyclist have to travel so that he covers 120 miles in 5 hours?

4. ( #6 /129)
   A car leaves a certain point and travels at the rate of 60 km/h. Another car leaves from the same point two hours later and travels the same route at 80 km/h. How long will it take the second car to catch the first car?

5. You run 21 miles in 180 minutes. What is your average speed in terms of miles per hour?
6. (8/130)
Frank and Mike live 22 miles apart. If each leaves his home at the same time and walks towards each other, Frank at a rate of six miles per hour and Mike at five miles per hour, how long will it take for them to meet?

7. (12/130)
A plane leaves New York bound for Houston traveling at the rate of 654 kilometers per hour. Thirty minutes later another plane leaves New York also bound for Houston traveling at the rate of 763 kilometers per hour. If both planes arrive at Houston at the same time, what is the distance between New York and Houston?

8. (13/131)
A man starts at a certain point and walks due east. One hour later another man starts at the same point and rides a bicycle due west, traveling nine miles per hour faster than the walker. After the cyclist has ridden for three hours they are 69 miles apart. What was the average rate of each?
9. (16/131)
   A stream is flowing at the rate of four kilometers per hour. A motorboat that can travel at thirty-two kilometers per hour in still water runs downstream. How long will it take to travel 99 kilometers.

10. (18/131)
    A plane flew for 3 ½ hours with a 30 mile per hour tailwind. It took five hours for the return trip. Find the speed of the plane in still air.

11. You start running with “your friend”. You are running at the rate of 7 1/2 miles per hour while your friend starts at 8 miles per hour. How long will it be before your friend is 880 yards ahead of you?

12. (20/132)
    A motorboat traveling at the rate of thirty kilometers per hour in still water takes four hours to go upstream and three hours to return. Find the rate of the current.
**Mixture Problems**

Although they sound different than the distance-rate-time problems we discussed – they are solved in a very similar way.

Examples:

1. (2/136)
   If peanuts sell for $2.50 per pound and cashews sell for $4.50 per pound, how many pounds of each must be used to produce 20 pounds of mixed nuts to sell for $3.75 per pound.

2. (4/136)
   A total of 35 coins in dimes and quarters has a value of $5.60 cents. How many of each type of coin are there?

3. (8/137)
   How many liters of pure alcohol must be added to ten liters of water to obtain a mixture of 60 % alcohol?
4.  (10/137)
A merchant mixes three brands of coffee that sell for $3.20, $3.80, and $4.50 per pound to produce 50 pounds of a blend that will sell for $3.69 per pound. If twice as much of the $3.20 brand is used as the $4.50 brand, how much of each brand is used?

5.  (12/137)
Mary has $4.25 in nickels, dimes, and quarters. She has twice as many dimes as quarters. If the total number of coins is 37, find the number of each kind of coin.

6.  (14/138)
How much water should be added to five liters of a 25% salt solution to obtain a 15% salt solution?

7.  A tank contains five liters of a 30% acid solution. How much must be drained off and replaced with a 50% acid solution to obtain five liters of 38% acid solution?
Chapter Four. 
Products and Factoring

Terminology: 
_terms, factors, coefficients, constants_

Recall that when you are given a literal expression the quantities being added are called terms while quantities being multiplied are called factors.

\[2xy\rightarrow\]
has one term \((2xy)\) but three factors \((2, x, \text{ and } y)\), 2 is called the coefficient

\[x + xy + 3\rightarrow\]
has three terms, the first term has 1 factor \((x)\), the second term has two factors \((x \text{ and } y)\)
the third term is the constant term or just “the constant”

We have found that it is useful to be able to write a number as a product of other numbers - prime factorization, GCF, LCM

The same is true for literal expressions.

Definition:
Factoring is the process of writing a literal expression from a sum or difference of terms to a product of factors.

We know that \(2x(x + y)\) consists of a product of three factors; \(2, x, \text{ and } x+y\)
If you distribute, you will get \(2x^2 + 2xy\). This means that \(2x^2 + 2xy = 2x(x + y)\rightarrow\) this is its factored form.

Note: we still have the same result just written in a different form.

We have discussed the following

A monomial: ___________________ A binomial: _______________ A trinomial: ____________

In general a polynomial: _______________________________

We also discussed the degree of polynomials.

For us, factoring will be taking a polynomial and writing as a product of other polynomials of equal or lesser degree. Before we begin, recall how to find the GCF of a group of numbers and then we will use the idea with polynomials.

\[\text{GCF}(20, 36) = \] \[\text{GCF}(8x, 12) = \] \[\text{GCF}(6xy, 8y) = \]
Factoring a polynomial by using the GCF

12 + 4x = ____________    y^2 − xy = ______________

2x − 12xy = ______________    6 − 12y = ______________

3x^2 - 6x = ______________    x^2 + 2x = ______________

What about

2x + 8xy + 8 = ________________    42x^2y - 12xy + 6y = ________________

In order to understand the next couple of methods – let’s review how to multiply polynomials.

Back in previous chapters we discussed how to multiply

1) a monomial by a monomial: had to do with rules of exponents

(2xy^2)(-3x^2y^3) = __________

2) a monomial times a polynomial: had to do with the distributive law

2x(x + xy + 4x^2) = ___________

3) a polynomial times a second polynomial – consider a very special case of polynomial
   (a binomial times a binomial)
   This involves the case that you may remember as the foil method

(x + 2y)(3x − 4y) = _________________

(2 + 3x)(5 − 3y) = _________________
Expand each of the following (multiply)

\[(a + 3)(a - 4)\]

= __________________________

\[4c + 2(a + 3)\]

= __________

\[(2x - 7)(2x + 7)\]

= _________________________

\[(a + b)^2\] = __________

\[(x + 2y)(x + y + 3xy)\] = _________________________

One of the cases above involved a product like

\[(3x + 2y)(3x - 2y)\]

What is special about this product

1) original problem: __________________________ 2) the result: __________________________

Try with

\[(5 - 2x)(5 + 2x)\] = __________________________

\[(3x + 4)(3x - 4)\] = __________________________

We call something of the form \((a - b)(a + b) = a^2 - b^2\) the difference of squares.

They are easy to recognize and once we see a few examples, they are easy to factor.

Find the product of \((1 - 4xy)(1 + 4xy)\). __________________________

Factor by recognizing that you have a difference of squares.

\[x^2 - 25\] = __________________________

\[4 - y^2\] = __________________________

\[9x^2 - 16\] = __________________________

\[12c^2 - 3d^2\] = __________________________
Is $9 + 25x^2$ a difference of squares? Why or why not?

Can you factor $12 - 3x^2$? If yes, go ahead and factor.

Is $x^9 - 9$ a difference of squares as we have done above?

Try to factor $x^6 - 36 = \ldots$ 

When factoring you should always check to see if there is a common factor — if there is, factor it.
Before we look at trinomials of the form \( ax^2 + bx + c \).

Consider the following methods of factoring; sum or difference of cubes and factoring by grouping

**Sum (or difference of cubes)**

Find the product: \((2 + x)(4 - 2x + x^2)\) =

\((x + 3y)(x^2 - 3xy + 9y^2)\) =

\((1 - xy)(1 + xy + x^2y^2)\) =

The result (right side) has the appearance of a difference (or sum) of cubes. The polynomials on the right side have to have a very special form.

**Factor each of the following**

\[ x^3 - 8 = \]

\[ 8x^3 + 27 = \]

\[ 1 + 64y^3 = \]

\[ 40x - 5x^4 = \]

**Take a second look at the GCF method.**

\[ 3 + 6x = \]

\[ xy + 2x = \]

\[ 3(W) + x(W) = \]

\[ 3(2x + y) + x(2x + y) = \]

\[ 9x(x - y) - y(x - y) = \]
Sometimes polynomial expression does not satisfy any of the methods we have looked at or even some that we have not
looked at. In that case it may be useful to group the polynomial and try to factor within the groups.

**Factor by Grouping**

1. \((2/159): \quad ax + ay + 4x + 4y = \) _________________

2. \((6/159): \quad 2ax + 3a + 4x + 6 = \) _________________

3. \((15/160): \quad 6ax - y + 2ay - 3x = \) _________________

Now we are ready to look at polynomials of the forms \(ax^2 + bx + c\). Sometimes they factor sometimes they do not

**Note:** If a polynomial does not factor by any of the known methods, we say the polynomial is prime.

First: Look at some special products.

**Perfect Squares:**

We call 1, 4, 9, 16, 25, … perfect squares. Why?

Is 144 a perfect square? It is if we can find an \(x\) so that \(x^2 = 144 \Rightarrow x = \) __________

We can describe polynomials as perfect squares.

What about polynomials? Is \(x^2\) a perfect square? __________ What about \(16x^2\)? __________

Is \(49x^4\)? __________ What about \(x^2 + 12x + 36\)? _______________
Look at some products to get an idea of what perfect squares look like when we talk about polynomials.

\[(x + y)^2 = \_] \quad (2 - 3x)^2 = \_
\]

\[(1 + 2xy)^2 = \_
\]

In general \((a + b)^2 = \_
\)

Polynomials of this form are common and are called perfect squares.

This gives us a nice quick way to square a binomial and it also provides a way to recognize a perfect square when written as a polynomial.

Find the product of

\[(2x + 3y)^2 = \_ \quad (1 - 2x)^2 = \_
\]

Factor each of the following by recognizing that the polynomial may be a perfect square.

\[x^2 + 16x + 64 = \_ \quad x^2 - 14x + 49 = \_
\]

\[4x^2 + 12x + 9 = \_ \quad x^2 + 12x + 25 = \_
\]

Complete each of the following so that the resulting polynomial represents a perfect square

\[x^2 + 20x + \_ \quad x^2 - 5x + \_
\]

\[x^2 + \_ + 81 \quad x^2 - \_ + 9/16\]
The last type of factoring we will attempt involves trinomials. Factor a polynomial of the form \( ax^2 + bx + c \). If it is a perfect square, we know what to do. Otherwise we have to use the method(s) that follow.

( May include the key method )

Think in terms of two cases.

Case 1. Suppose that \( a = 1 \) (coefficient of \( x^2 \)). We want to factor is \( x^2 + bx + c \).

Most times the polynomial will not factor but when it does it will look like \( (x + e)(x + g) \) the only thing we have to know is the value of \( e \) and \( g \). They come from the factors of \( c \).

Ex. Factor \( x^2 + 10x + 9 \). \( \rightarrow (x + ____)(x + ____), \) the only factors of 9 are 1 and 9.

If it factors, it will have to be \( \rightarrow x^2 + 8x + 9 = (x + 9)(x + 1) \). Is this the answer?

Try these

\[
\begin{align*}
\text{x}^2 + 12x + 32 &= \_\_\_\_\_\_\_\_\_ \\
x^2 - 8x + 12 &= \_\_\_\_\_\_\_\_\_ \\
x^2 + 4x - 12 &= \_\_\_\_\_\_\_\_\_ \\
x^2 - 6x - 40 &= \_\_\_\_\_\_\_\_\_ \\
x^2 + 2x - 8 &= \_\_\_\_\_\_\_\_\_ \\
\end{align*}
\]
More difficult examples:

\[
\begin{align*}
\frac{50}{163} & \quad \frac{44}{163} \\
\left( x^2 + 23x - 140 \right) & \quad \left( x^2 - 39x + 140 \right)
\end{align*}
\]

Case II. What if \( a \) is not necessarily equal to 1? (It could be, but it does not have to be)

We want to factor \( ax^2 + bx + c \). If it factors, it will have the form \((dx + e)(fx + g)\).

Now we have two find two additional values \( d \) and \( f \). They come from the factors of \( a \).

ex. Factor \((2/165)\)

\[
2x^2 + 11x + 5; \quad \text{the factors of 2 are 2 and 1} \rightarrow \text{these make up the coefficients of } x. \quad \text{No other choice is available.}
\]

\[
2x^2 + 11x + 5 = (2x + \underline{\quad}) (x + \underline{\quad}) \quad \text{work as in case 1. The factors of 5 are 1 and 5.}
\]

Here, we have to worry whether we use 1 and 5 or 5 and 1. Trial and error will tell us which one works – if any.

Ex. Factor \((16/165)\)

\[
3x^2 - 13x - 10 = \underline{\quad}
\]

Ex. Factor \((9/165)\)

\[
2x^2 + 3x - 20 = \underline{\quad}
\]

Ex. Factor \((34/166)\)

\[
6x^2 - 5x - 6 = \underline{\quad}
\]
Your author discusses what he calls the “key number”. Instead of trial and error as we did in the preceding example – we use a little bit of order. Why don’t we call this the “key number method”

**Key Number Method**

We are still trying to factor polynomials of the form $ax^2 + bx + c$.

Define the key number as the product of the coefficient of the 1st term (a) and the constant (c).

key number = $ac$ ($a \cdot c$).

examples:

$3x^2 - 4x + 10 \rightarrow$ key number = $3(10) = 30$

$5x^2 - 2x - 3 \rightarrow$ key number = $(5)(-3) = -15$

Your author discusses one way to use the key number on page 176 in your text. I will discuss the second way (on page 178 in your text).

**This method uses the key number and grouping to factor our trinomials**

Factor

1. (2/178) $x^2 - x - 6$ (you obviously do not need the key method – but let’s start with an easy one)

   key number = $1((-6)) = -6$

   Find the factors of -6 (includes the sign) so that their “sum” will be equal to the coefficient of the middle term.

   middle term = $-x$, coefficient = $-1$

   factors of -6 = $(-2)(3)$, $(2)(-3)$, $(1)(-6)$, and $(1)(6)$. Four possible choices

   Sum must equal to -1.

   $(2)(-3)$ are it.

   Break up the middle term to use these numbers we just found

   $x^2 - x - 6 = x^2 + 2x - 3x - 6 = (x^2 + 2x) + (-3x - 6)$, we grouped them - now factor.

   $= x(x + 2) - 3(x + 2) = (x + 2)(x - 3)$. Done.

2. Let’s factor (8/178) $2x^2 - 11x + 15$

3. Try $16x^2 - 8x - 3$. 

61
**Long Division of polynomials**

Recall long division with just numbers

\[
234 \div 4 \rightarrow
\]

\[
8324 \div 12 \rightarrow
\]

Notice: \(24 \div 4 = 6\), we say that 4 is a factor of 24. There was no remainder in the division.

Is 9 a factor of 28? ___________ Why? ____________________________

Try it with polynomials.

Is \(x + 2\) a factor of \(x^3 - x^2 - 2x + 2\)?

One way would be to try to factor \(x^3 - x^2 - 2x + 2\) in most cases factoring is not obvious.

Try long division and see if there is a remainder.

Is \(x - 1\) an exact divisor of \(x^4 + 2x - 3\)?
You were told that 17 was a factor of 629. Find the other(s) factor(s) of 629. How would you find it (them)?

If $3x + 2$ is a factor of $3x^3 + 11x^2 - 9x - 10$, find the other factor(s).

More Factoring:

$x^4 - 16 = \text{________________________} \hspace{1cm} x^4 + 2x^2 - 3 = \text{________________________}

x^2 (x - 3) - 4(x - 3) = \text{________________________}
Chapter Test and Review. Begins on page 183 and ends on page 190. There is a cumulative test covers all first four chapters.
Additional Problems. - not part of notes – but use them if they are of any use to you.

Factor by grouping

factor: \(2x \cdot y - 3y = \) \[\text{__________}\] 
\(2x(y + 1) - 3(y + 1) = \) \[\text{__________}\]
\(2a(a + 1) - 3(a + 1) = \) \[\text{__________}\] 
\(x(2y - 3) + (2y - 3) = \) \[\text{__________}\]

Also,
\(ax + ay + 4x + 4y = \) \[\text{__________}\] 
\(3ax + 5a + 6x + 10 = \) \[\text{__________}\]

Difference of Squares –

Product: Multiply each of the following polynomials – try to see a pattern
\((x + y)(x - y) = \) \[\text{__________}\] 
\((2x - y)(2x + y) = \) \[\text{__________}\]
\((3x + 4y)(3x - 4y) = \) \[\text{__________}\] 
\((x - 5y)(x + 5y) = \) \[\text{__________}\]

Multiply – \((2x + 7y)(2x - 7y) = \) \[\text{__________}\]

Factor each of the following polynomials.

a) \(x^2 - 16 = \) \[\text{__________}\] 
\(x^2 - 49y^2 = \) \[\text{__________}\]

c) \(25 - y^2 = \) \[\text{__________}\] 
\(2x - 2x^3 = \) \[\text{__________}\]

e) \(5 - 40x^2 = \) \[\text{__________}\] 
\(x^2 + y^2 = \) \[\text{__________}\]

cubes: sum – difference
1. Multiply each of the following. Do not factor.

\[ 2x ( x + 3 ) = \] 
\[ xy ( 2x - 3y ) = \] 
\[ ( x - 2y )^2 = \]

2. Factor each of the following polynomials.

\[ 3x - 12xy = \] 
\[ 8 - 4x = \] 
\[ xy + 2xy = \]

3. One number is four more than a second number and eight less than a third number. Find the three numbers if their sum is 49.

4. A student purchases a calculator for $24.50. Another student purchases two notebooks and a calculator for $37.00. Find the price of the calculator and the price of the notebook.