Math 1321
Brief Review of Algebra

To solve equations of the form \( ax^2 + bx + c = 0 \), you can either solve by factoring, completing the square, or by using the quadratic formula.

**Quadratic formula:**

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

ex. \( x^2 + 2x = 0 \)  

ex. \( (x + 2)^2 = 9 \)  

ex. \( x^2 - 5x - 6 = 0 \)

**Factoring:**

- **Difference of squares:** \( a^2x^2 - b^2y^2 \)
  
multiply: \( (2x - 3y)(2x + 3y) = \) 
  
  factor: \( x^2 - 16 = \) 
  
solve the following equation: \( 4x^3 - 9x = 0 \) 

- **Difference (sum) of cubes**
  
multiply: \( (2x + y)(4x^2 - 2xy + y^2) = \) 
  
  factor: \( x^3 + 8y^3 = \) 
  
solve: \( x^3 - 27 = 0 \) 

- **Trinomials**
  
multiply: \( (3x + 4y)(2x - y) = \) 
  
  factor: \( x^2 - 4x - 12 = \) 
  
  Solve: \( x^2 - 12x + 20 = \)
Completing the Square
Not a preferred way to solve an equation – but a useful one.

---- you want to take a polynomial equation of the form

\[ ax^2 + bx + c = 0 \]

and change it to a polynomial equation of the form \((x + d)^2 = k\)

\[ 2x^2 + 5x + 3 = 0 \]

Solving equations:

\[
\frac{x - 1}{2} = \frac{x}{2} \quad x - 2(1 - 2x) = 3 \quad \frac{3}{x - 1} = \frac{2}{x + 1}
\]

\[
\sin x = \frac{1}{2} \quad \log_4 x + \log_4 (x + 3) = 1 \quad 2^x = 5
\]
Graphs
you should be familiar with as many of the following as possible.

Lines:  \( ax + by = c \) (include vertical, horizontal, slant) → we will review some of it
Parabolas:  \( y = ax^2 + bx + c = y \) (open up or down – need to be able to find vertex – we will study them in detail.
Others: absolute values, trig. functions, inverse trig. functions, logarithms, exponential functions

Graphs:

1) \( y = 2x - 3 \)
2) \( y = x^2 + 2x \)
3) \( y = |x| \)
4) \( y = 2^x \)
5) \( x = \sin y \)
6) \( y^2 = x^2 \)

Intercepts: x and y -intercepts

Find the

x-intercepts: \( y = x^2 - 2x - 3 \) → x-intercepts = _________

\( x = \sin y \) → x - intercepts = _________

\( y = 2^x \) → x - intercepts = _________

y -intercepts: \( y = |x + 2| \) → y - intercepts = ____________

\( y = \log_3 x \) → y - intercepts = ____________

\( y^2 = x^2 - 4 \) → y -intercepts = ____________
Geometry:
Pythagorean Thm. : Given a right triangle with sides a, b, c where a and b represent the legs (two smaller sides) and c is the hypotenuse, then $a^2 + b^2 = c^2$

ex. Find the missing side in each of the following cases.

ex. A 12 ft. ladder leans into a building creating a right triangle in which one of the acute angles is equal to 45°. What is the exact length of the other two sides.

ex. Complete the sides of the given right triangles. (30°, 60°, 90°; and the 45°, 45°, 90° right triangles)

angle A = 45°, angle B = 45°, angle C = 90°

angle A = 30°, angle B = 60°, angle C = 90°
**Geometric Figures**
Definitions of each of the basic geometric figures such as squares, rectangles, triangles, right triangles, isosceles triangles, equilateral triangles, quadrilaterals, parallelograms, trapezoids, rhombus

**Quadrilaterals:** a quadrilateral: four sided figure
1. a trapezoid: a quadrilateral with one pair of opposite sides parallel
2. a parallelogram: a quadrilateral with opposite sides parallel
3. a rhombus: a parallelogram with all sides of equal length
4. a rectangle: a parallelogram with adjacent sides being perpendicular
5. a square: a rectangle with all sides of equal length

**Triangles:**
1. acute: Triangle with __________________________
2. obtuse: triangle with ____________________________
3. right: triangle with ____________________________
4. scalene: triangle with __________________________
5. isosceles: triangle with _________________________
6. equilateral: triangle with ______________________

**Congruent(triangles):**
Corresponding angles are equal (congruent), corresponding sides are congruent (figures look the same and are of the same size – copies that have been rotated, flipped, ...

![Congruent Triangles](image)

**Similar figures(triangles):**
Corresponding angles are equal (congruent) - figures look the same but may be of different sizes sides of triangle may not be congruent. There is a scaling factor, r, to go from one triangle to another.

![Similar Triangles](image)

ht = h

What is the ht? _____

ht = \( \frac{2}{3} h \)
**Property:** (of congruent and similar triangles) Given triangles ABC and DEF with heights h and k

1) scaling factor:

\[ \frac{12}{36} \]

\[ h = 5 \]

Perimeter ? \( \) \( \) \( k = ? \) \( \) \( \) \( \) Areas ? \( \)

2) scaling factor (ht):

\[ \text{base} = 20 \]

\[ \text{ht} = 12 \text{ inches} \]

\[ \text{Perimeter} \ P \]

\[ \text{Area} = \] \( \)

\[ \text{base} = ? \]

\[ \text{ht} = 30 \text{ inches} \]

\[ \text{Perimeter} = ? \] \( \)

\[ \text{Area} = \] \( \)

Which one implies the other? congruent \( \Rightarrow \) similar or similar \( \Rightarrow \) congruent?

Use of triangles:

ex. Length of the shadow of a 6 ft man created by a light on a 12 foot post that is 8 feet away from the man.
A ladder leans against a wall touching a window that is 12 ft high. The ladder must make an angle that does not exceed 30° with the ground for safety purposes. What must the minimum length of the ladder?

**Def. Absolute Value**

Let \( c \) be a real number. The absolute value of \( c \), written \(|c|\) is defined by

\[
|c| = \begin{cases} 
  c & \text{if } c \geq 0 \\
  -(c) & \text{if } c < 0
\end{cases}
\]

In other words, regardless of the value of \( c \), \(|c|\) will equal only one value → either \(|c| = c\) or \(|c| = -c\)

(Choose the one that gives you a positive result → but these are your only choice; the number or its opposite)

What is the opposite of \(-2 + \pi\)?

So the two possible guesses as to \(|-2 + \pi|\) are? _________ or _________ Which one is the right one?

**ex. Find each of the following absolute values.**

1. \(|-4/3| = ________ \) \(|x^2| = ________ \) \(-|x| = _____\) if \( x \) is a whole number.

2. \(|3 - \sqrt{13}| = ________ \)

3. \(|1 - x^2| = ________\) if \( x \) is a nonzero integer.

4. \(\frac{x}{|x|} = ________\), if \( x \) is a natural number.

5. \(\frac{1 - |x|}{x - 1} = ________\) \(x \neq 1\)

6. \(|x^2| = ________\)
Properties.

1. $|c| \geq 0$ and the only time it equals zero is when $c = 0$

2. $|c|^2 = (c)^2$

3. $\sqrt{c^2} = \quad$ (the square root of $c^2$)

4. $|c|^2 = (\bar{c}c)$, where $\bar{c}$ represents the conjugate of $c$

5. $|a + b| \leq |a| + |b|$ (triangular inequality)

6. $|a \cdot b| = |a| \cdot |b|$

7. $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$

Distances

There are two types of distances that are of interest to us;
**directed distance** (direction matters) which can be positive or negative (or zero), and

$$AB \neq BA$$

**undirected distance**, which is always nonnegative.

$$AB = BA$$

In our drawing assume that if we move in the direction of the arrow distance will be considered positive, against it we consider the distance to be negative.

IF the directed distance from $A$ to $B = 7$, then the directed distance from $B$ to $A$ would be $\quad$ (enter value)

the undirected distance from $B$ to $A$ (or from $A$ to $B$) is equal to $\quad$ (enter value)
We use a number line to represent real numbers. We say there is a 1-1 relationship between the points on the line and the real numbers by thinking of the numbers as a distance (directed) from the origin.

We say that $AB$ and $BA$ are numerically equal — same numbers but possibly different signs.
Notice that $AB = -BA$, or $|AB| = |BA|$.

We use this type of directed line (directed distance) to represent the real numbers on a graph; the real number line.

Thm 1. Let $A$, $B$, and $C$ be any three points on a line (collinear points), then

$AB + BC = AC$ and $AC + CB = AB$ and $BA + AC = BC$

Pf.

Suppose that $A$, $B$, and $C$ are collinear points in the following order
Suppose we want the solution of the equation

\[ x^2 - 4x - 5 = 0 \]

we usually write \{ _______ \} or \{ _______ \}

But what about the solution of \( y = 2x + 2 \) \( (a \text{ solution}) \) \______________

Is \((2, 6)\) a solution? What about \((6, 2)\)? What can you conclude from this? \______________

**Def. ordered pairs of numbers**

The pair \((x, y)\) is called an ordered pair of numbers if \((x, y) = (y, x)\) if and only if \((\text{Iff}) x = y\).

\[ (2, 3) \neq (3, 2) \text{ while } (5, 5) = (5, 5) \]

**Def.** Let \(A = \{ a, b, c \}\) and \(B = \{ 1, 2 \}\). We define the cross product of \(A\) and \(B\), \(A \times B\) (Cartesian Product) as the set of all ordered pairs \((a, b)\) such that \(a\) is from \(A\) and \(b\) is from \(B\)

\[ \text{i.e., } A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \} \]

In this case \(A \times B = \) ___________________________________________________________________________
Rectangular Coordinate System: (Cartesian Coordinate System)

If we let A and B equal infinite sets (like say - the set of real numbers) we generate a plane – a rectangular coordinate system. (the cross product of the infinite sets of real numbers and another copy of the set of real numbers)

$A \times B = \mathbb{R} \times \mathbb{R} = \text{an infinite collection of ordered pairs of real numbers} – \text{in this case they represent Plane:}$

$x: \text{called the } x\text{-coordinate, abscissa: distance from the } y\text{-axis to the point (directed distance)}$

$y: \text{called the } y\text{-coordinate, ordinate: distance from the } x\text{-axis to the point (directed distance)}$

ex. (-3, -1) and (4, 2) - Plot these points and find the distance between them.
Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Case 1: Both on the same horizontal line →
Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on a horizontal line. → distance _______________

Case 2: Both on the same vertical line →
Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on a vertical line → distance _______________

In general, we come up with following formula. Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ what is the distance between them (undirected distance).

**Thm 2.** Let $A(x_1, y_1)$ and $B(x_2, y_2)$ represent any two points in the plane. The distance between $A$ and $B$ (undirected distance) is represented by

$$d = \________________________$$

Pf.
Use the distance formula to answer the following questions.

1. Are the following three points collinear? A( -3, 1 ), B( 1, 3 ), C( 10, 8 )
   Property of a triangles: sum of any two sides > third side

2. Show that the following points form an equilateral triangle. \( A(-2,0), B(2,0), \text{ and } C(0,2\sqrt{3}) \)

3. If \((x, 4)\) is equidistant from \(P(2, 6)\) and \(Q(7, -2)\), then find \(x\) ________

4. Find the point on the y-axis that is equidistant from \(P(-4, -2)\) and \(Q(3, 1)\)

5. Find the area of a triangle with vertices \(A(3, 1), B(4, 2), C(-2, 4)\)
There is an alternate way to find an answer to question #5 but to do this we have recall DETERMINANTS.

ex. Given matrices A, B, C find det A, det B, det C

$$A = \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 3 \\ 4 & -4 & 5 \end{bmatrix}$$

det A = _____________

det B = _____________

det C = ________________

Property: Given that points ABC form a triangle find its area.

A ( -2, 4 ) ,  B ( -1, 5 ) ,  C ( 4, 2 )

area of triangle = _______

ex. Use #38 on page 12 in your book and determinants to find the area of the given //gram.

A ( -2, 4 ), B ( -1, 5 ), C(4, 2), D ( 3, 1 )
Recall
1) relationship between radians and degrees:

\[ 2\pi \text{ radians} = 360^\circ \implies \pi \text{ radians} = 180^\circ \implies \] We can convert from one form to the other.

You can use dimensional units to convert from one form to another.
\[ \frac{\pi}{180} \text{ or } \frac{180}{\pi} \text{ as needed} \]

Basic Relationships – should keep in mind

\[ 180^\circ = \underline{\quad} \text{ radians} \quad \quad 90^\circ = \underline{\quad} \text{ radians} \quad \quad 360^\circ = \underline{\quad} \text{ radians} \]

\[ 45^\circ = \underline{\quad} \quad 30^\circ = \underline{\quad} \quad 60^\circ = \underline{\quad} \quad 270^\circ = \underline{\quad} \]

Additional examples:

\[ 260^\circ \rightarrow \underline{\quad} \text{ radians} \quad \quad 2\pi/15 = \underline{\quad} \text{ degrees} \]

1 radian = \underline{\quad} degrees

True or False.

\[ \underline{\quad} \text{ 1) } \pi \text{ radians} = 180^\circ \quad \underline{\quad} \text{ 2) } \pi \text{ is always equal to } 180^\circ \]

\[ \underline{\quad} \text{ 3) } 1 \text{ revolution} = \pi \text{ radians} \quad \underline{\quad} \text{ 4) } \]

2) Transverse line (of two parallel lines) - transversal

If two angles look the same, they are.

In the following picture \( L_1 \) and \( L_2 \) are parallel and line \( T \) represents the transversal line.
3) **From Trig:**

Recall definition of tangent function – \( \tan \theta = \frac{\text{opposite}}{\text{adjacent}} \)

as well as the other two main trig. functions → \( \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \), \( \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \)

Also, \( \tan (A - B) = ? \) → __________________  \( \tan (180^\circ - A) = \) ________________

4) **From Trig:** recall the special triangles – 30º-60º right triangle and the 45º-45º right triangle

![30º-60º right triangle](image1)

![45º-45º right triangle](image2)

**Def.**

Let \( L \) be a horizontal line. Define its angle of inclination as 0º.

**Def.**

Let \( L \) be a line that crosses the x-axis in at most one point (non-horizontal line) and let \( \theta \) be the angle that \( L \) makes with the positive side (direction) of the x-axis with 0º ≤ \( \theta \) < 180º. We call \( \theta \) the inclination or the angle of inclination of \( L \).
Note: If two lines are parallel, then they have the same angle of inclination.
Conversely, if two lines have the same angle of inclination, then they are parallel lines.

Def. of slope of a line.
The tangent of the angle of inclination, \( \theta \), of a line is called the slope of the line.

\[
tan \theta = \text{slope} = m = \frac{\text{rise}}{\text{run}}
\]

ex. Let \( L \) be a line with \( \theta = 30^\circ \) find \( \tan \theta = \) ____________

ex. Let \( L \) be a line with \( \theta = 145^\circ \), find its slope \( m = \) ____________

ex. Suppose that \( L \) is a line with slope 2 \( m = 2 \) find its angle of inclination to the nearest degree _______________

ex. If \( L \) is a line with slope = -4, find its angle of inclination to the nearest degree. ________________

Thm 3.
If a line \( L \) is given with slope \( m \) and it passes through the points \( A(x_1, y_1) \) and \( B(x_2, y_2) \), then the slope of \( L \) is given by

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Pf.
ex. **Use slopes** to prove or disprove that the following points are co-linear 

A(1, 6), B(3, 14), and C(-2, -2)

ex.
Given three vertices of a triangle can we find the angles of the triangle; A(2, -1), B(4, 2) and C(-1, 2)

ex. Let $L_1$ and $L_2$ be two given lines with $L_1$ of slope $\frac{3}{4}$ and $L_2$ of slope $(-1/2)$.
What is the acute angle created at the intersection of the lines

In general we can use the following formula to find an angle $\phi$ at the intersection of any two line or line segments.

Construction: Let $\phi$ be the angle we are seeking and let $\theta_1$ and $\theta_2$ be the corresponding angles of inclination.
Thm4: Let \( L_1 \) and \( L_2 \) be two given lines with known slopes.

a) If \( L_1 \) and \( L_2 \) are parallel, then they have equal slopes

b) If \( L_1 \) and \( L_2 \) are perpendicular, then their slopes are negative reciprocals of each other, i.e., the product of their slopes equals -1.

ex. Show that the following three points are the vertices of a right triangle (-1, -1), (16, -1), (0, 3)

ex. Show that the following vertices of a parallelogram form a rectangle. (5, 7), (1, 1), (4, -1), (8, 5)

ex. How would you show that a given quadrilateral it is a parallelogram?
Division of a line segment.

Take a line segment with endpoints \( A(x_1, y_1) \) and \( B(x_2, y_2) \) (special case with \( A(4, -1) \) and \( B(-4, 3) \))

1) I want a point on the segment that is equidistant from the endpoints (midpoint):

2) What if you want a special point on the segment – that is not necessarily at the midpoint? (Say, 2/5ths of the way from point A to point B.

3) What if you want a point(s) that is twice as far from A than from B (not necessarily on the segment)?

What about on the segment \( AB \)? Is there only one?
Formula: (directed distance from A to B)

to find a point P(x,y) that is on the segment AB (extended through A or B)

\[ x = x_1 + r(x_2 - x_1) \text{ where } r = \frac{AP}{AB} \]
\[ y = y_1 + r(y_2 - y_1) \text{, where } r = \frac{AP}{AB} \]

ex. If a person stands underneath a ladder that is 10 feet long and the person is six feet tall. How far above the ground does the ladder touch the wall if the person is 2/5 of the way from foot of the ladder to the wall?
36/28
A 30 lb child is sitting at A(2,3) and a 50 lb child is sitting at B(12, 7), where units are feet. Find the point P between A and B which could be used as the fulcrum of a teeterboard putting the two children in equilibrium. (Hint: 30AP = 50PB or AP/PB = 50/30)

38/28
A person 6 feet tall is standing near a street light so that he is 4/10 of the distance from the pole to the tip of his shadow. How high above the ground is the light bulb? If the person’s head is exactly 5 feet from the light bulb, how far is the person from the pole, and how long is the shadow?

39/29
A backpacker 6 ft tall sees the peak of a mountain reflected in a small calm pool. The pool is 2 miles from the peak, according to a map. If the backpacker is 30 ft from the point of reflection in the pool, how high is the peak above the level of the pool?
Analytic Proofs: use slopes, distances, geometric properties of figure, rectangular coordinate system.

ex. Prove that the diagonals of a parallelogram bisect each other.

We want to write these statements in the form \textit{IF A, Then B.}  
A: will represent the statement that we can use, the statements that are assumed to be true  
B: will represent the statement that we will try to prove

ex. Prove that a parallelogram whose diagonals are perpendicular is a rhombus.

A:  
B:

What is the area of a rhombus if you are told that the diagonals are of length d?
Prove
If the diagonals of a rectangle are of equal length, the figure is a rectangle.

Prove
The diagonals of a rhombus are perpendicular
Relations and Functions

Def. A relation is a subset of \( \mathbb{R} \times \mathbb{R} \). ----a set of ordered pairs of numbers.

The set of all first elements in a relation is called the **domain** of the relation.

The set of all 2nd elements is called the **range** of the relation.

\[
\begin{align*}
R_1 & : 1 \rightarrow 5, 2 \rightarrow 8, 3 \rightarrow 11 \\
R_2 & : 9 \rightarrow -4, 16 \rightarrow -3, 3 \rightarrow 4 \\
R_3 & : 90 \rightarrow 4, 85 \rightarrow 3, 95 \rightarrow 2, 80 \rightarrow 3, 75 \rightarrow 2
\end{align*}
\]

Rule: add two to twice the number

Domain: __________

Rule: ?

Domain: __________

Rule: ?

Range: __________

Write each of these relations in terms of ordered pairs.

\[
\begin{align*}
R_1: & \quad (1, 5), (2, 8), (3, 11) \\
R_2: & \quad (9, -4), (16, -3), (3, 4) \\
R_3: & \quad (90, 4), (85, 3), (95, 2), (80, 3), (75, 2)
\end{align*}
\]

Def. A **function** is a relation in which no two ordered pairs have the same first element and different second elements. (i.e. one x will generate a single value of y)

\[
\begin{align*}
y & = 2x - 3 \\
y & = x^2 + 2
\end{align*}
\]

Function or Relation

Domain: ________________

Range: ________________

Function or Relation

Domain: ________________

Range: ________________
ex. circle of radius 2 with center at the origin

Function or Relation

Domain: ___________________________

Range: ________________

ex. \( x = y^2 + 2 \)

Function or Relation

Domain: ___________________________

Range: ________________

ex. \( x = \sin y \)

Function or Relation

Domain: ___________________________

Range: ________________

ex. \( y = \frac{e^x - e^{-x}}{2} \)

Function or Relation

Domain: ___________________________

Range: ________________

ex. \( x^2 + 4y^2 = 4 \)

Function or Relation

Domain: ___________________________

Range: ________________

Note:

Vertical Line Test – a way to determine if a relation is also a function: every vertical line must cross the graph of the relation in at most one point.

notation:

function: \( y = 2x + 3 \) or we can write \( f(x) = 2x + 3 \)

ex. find \( y \) if \( x = 2 \) ==> we write \( f(2) = \) ? ________________

Note: \( f(0) \) represents the y-intercept.
Def.
Let \( f(x) \) be a function. We say that \( f(x) \) is even provided \( f(-x) = f(x) \).
If \( f(-x) = -f(x) \), we say that \( f(x) \) is odd.
If neither happens, then \( f(x) \) is said to be neither odd nor even.

Show whether the following are even, odd, or neither

ex. \( f(x) = 4x^3 \)  

b) \( f(x) = -2x \)

ex. \( f(x) = 2x + 6 \)  

b) what about \( f(x) = \sin x \)

More examples on even, odd, neither

ex. \( f(x) = \frac{e^x + e^{-x}}{2} \)  

ex. \( f(x) = \arccos x \)
Given a graph – can you determine if the function is even, odd, or neither?

**even**: if \( f(-x) = f(x) \)

**odd**: if \( f(-x) = -f(x) \)
**Graphs of an equation** –

We can’t always write all the solutions of an equation but a graph will indicate all such solutions.

Depending on the type of curve, it may be useful to know the x and y – intercepts, properties of the graph, slopes.

ex. \( f(x) = 2x \)  \hspace{1cm} \text{or} \hspace{1cm} f(x) = x^2 + 1

ex. \( f(x) = |x - 2| \)

ex. \( y = \log_2 x \)

ex. \( y = \cos x \)

ex. \( x = \tan y \)
Equation of a graph

given some geometric property, we may be able to find a set of points (a relation) that represents the graph in an equation form

1) Find the set of all points that are twice as far from A(-2,1) as from B(6,-5)

2) Find the set of points P(x,y) so that P forms a right triangle with the vertices A(-4,0) and B(4,0) where P is the vertex of the right angle.
We have looked at the idea of a function being even, odd, or neither.

ex. Pick a number from 1 – 10: say x
Add two to that number: f(x) = x + 2  
square a number:  say g(x) = x^2

Put these two statements together:
add two to that number and square the result. Rather than using two functions to
calculate the result, can we use one statement (one function) to do both

h(x) = ________________________

Def.
(Composition of Functions) Let f(x) and g(x) be given functions. Define h(x) = f( g( x ) )
We call h(x) a composition of functions. We write h(x) = (f o g ) (x)

We can also change the order and get a different function h(x) = g o f (x) = g ( f ( x ) ) .

ex. Let f(x) = 2 – 3x and g(x) = 2^x. Find f o g ( 2 ) = ___________ and g o f ( 2 ) = ___________

ex. Let f(x) = 3x + 2 and g(x) = \( \frac{x - 2}{3} \). Find fog(8) = ___________ gof(8). ___________

ex. Let f(x) = 3^x and g(x) = \log_3 x . Find fog(9) ________________, gof(1). ____________
Prove the following statements.

a) Suppose that \( f(x) \) is odd and \( g(x) \) is odd. Prove that the sum of two odd functions is ________________

b) Prove that the sum of two even functions is ________________

c) Prove that the sum of an even and an odd function is ________________

d) Prove that the product of two odd functions is ________________
e) Prove that the Product of an odd and an even function is ___________

More on composition functions:
Prove that
a) If \( f(x) \) is even and \( g(x) \) is even, then \( f \circ g \) is even

b) If \( f(x) \) is odd and \( g(x) \) is odd, then \( f \circ g \) is _______

c) If \( f(x) \) is odd and \( g(x) \) is even, then \( f \circ g \) is _______________ what about \( g \circ f \)? _______________
Chapter 2
Lines and First Degree Equations

Thm. Let $L$ be a line passing through the point $P(x_1, y_1)$

1) If $L$ is vertical, the equation of $L$ is given by _______________ and it has no slope (undefined slope).

2) If $L$ is not vertical, then it has a slope, say $m$, and it has a y-intercept, say $b$. The equation of $L$ is given by _______________

Pf.

A linear equation of the form $Ax + By + C = 0$ can be written in one of the two forms below

1) $x = -C/A$ if the line has an undefined slope
2) $y = -\frac{A}{B} x - \frac{C}{B}$

ex. Find the equation of the line that passes through the point $(2, -1)$ and has slope 4.

Slope of a line -
1) if vertical $\Rightarrow$ no slope
2) if horizontal $\Rightarrow$ slope is zero
3) otherwise $\Rightarrow$ $m = \frac{y_2 - y_1}{x_2 - x_1}$

Equations of a line - passing through the point $(a, b)$

1) if vertical $\Rightarrow$ __________
2) if horizontal $\Rightarrow$ __________

ex. Find the equation of the line
a) is vertical and passes through the point $(2, -4) \Rightarrow$ _______________

b) is parallel to the x-axis and passes through the point $(3, -2) \Rightarrow$ __________

In general we use the slope intercept form $\Rightarrow y = mx + b$ or one of the following forms to find the equation of a line
Other Forms of a line

1) intercept form

2) point-slope form

3) Standard form of a line

ex. Find the equation of the perpendicular bisector of the segment AB; A(2, -1) and B( - 4, - 5 )

ex. Given $Ax + By = C$, find the equation of the line that is

   a) parallel to it and passes through the point $(x_1, y_1 )$

   b) perpendicular to it and passes through the point $(x_2, y_2 )$

ex. Find the equation of the line that is

   a) parallel to the line $2x – y = 4$ and passes through the point ( 4, - 1) .

   $_____________
b) Find the equation of the line that is perpendicular to the line $x - 2y = 3$ and passes through the origin.

\[
\]

\[
\]

c) Find the equation of the line that has slope $m=-2$ and passes through the point $(4, -1)$

Find the equation of a line that

1) Is tangent to a circle $C: x^2 + (y-1)^2 = 10$ at the point $(3, 2)$

2) Is a distance of 3 units from the line $x + 2y = 0$. (HINT: we must be talking about a parallel line $\rightarrow ?$) (nice short way coming up but for now see if we can be creative – suggestions ?)

3) 54/58:
   When does the Celsius reading and the Fahrenheit reading be the same number (reads and feels the same)

4) 59/59

Show that the equation of the line through $(x_1, y_1)$ and $(x_2, y_2)$ is given by

\[
\begin{vmatrix}
  x & y & 1 \\
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
\end{vmatrix}
\]
5) Show that if $Ax + By = C$ and $Dx + Ey = F$ are parallel lines, then $A/B = D/E$.

Definition:
A function $f$ is increasing on the open interval $(a, b)$ in its domain if for every pair of $x$’s; $x_1$ and $x_2$ of $(a, b)$ if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

Write a similar definition for increasing.

text examples:
Show by using the definitions above (not any graph) that the function $f(x) = -4x + 3$ is decreasing on the set of real numbers.

Show that $f(x) = ax^2 + bx$, where $a, b > 0$, is decreasing on the interval $(-\infty, 0)$.
Systems of linear Equations.
Recall: elimination, substitution method

a) \[2x - y = 6\]
\[3x + 2y = 2\]

b) \[4x + 6y = 1\]
\[6x + 9y = 3\]

c) \[2x + y = 3\]
\[x - 3y = 2\]

d) \[2x - 6y = 8\]
\[3x - 9y = 12\]

Applications:

ex. Break-even analysis –
Suppose that it costs $4000 / month to set and produce a monthly magazine plus an additional
cost of $2.50 per magazine. If the magazine is to be sold for $6.50, then how many magazines
must be sold per month to break even.

ex. If the price of an item is at \( p \), then the demand of the item is given by \( d = 400 - 0.01p \)
If the price of an item is at \( p \), then the supply amount is determined by \( s = 300 + 0.15p \)
At what price will the supply and demand be equal – equilibrium point.
We have talked about distance between two points (directed and undirected). We can also define directed distance from a line to a point. This was the original definition of the coordinates of a point P(x,y).

- x: the directed distance from the y-axis; y: the directed distance from the x-axis.

**Directed distance from a line to a point:**
from a line \( ax + by + c = 0 \) to a point \( P(x,y) \)

We have defined the undirected distance between two points: (distance formula)

**What about the directed distance from a line to a point?**

begin with the distance from \( x = 4 \) to a point \( (2, -3) \)  

what about from \( y = 4 \) to the point \( (10, 6) \)

Define the distance to be positive if the point is ________________________________
In general, what do we do if the line is slant? Write the line in the form $ax + by + c = 0$ and the point $P(x_1, y_1)$, then we get the directed distance from the line to the point is represented by the following formula.

$$d = \text{ }
$$

**Outline of the construction of the formula.**

1) Draw the given line $L$ and construct a second line $L_2$ parallel to $L$ passing through the given point.

Find a third line (label it $L_3$) that is perpendicular to $L$ and $L_2$. Find the distance from $L$ to $L_2$ can now be done in terms of three points, one being the origin.

$$O(0,0), \quad P\left(\frac{-AC}{A^2 + B^2}, \frac{-BC}{A^2 + B^2}\right), \quad Q\left(\frac{-AC'}{A^2 + B^2}, \frac{-BC'}{A^2 + B^2}\right)$$

Where do these three points come from?

ex. Find the distance from the line $x = 4$ to the point (-2, 3).

ex. Find the distance from the line $4x + 3y = 1$ to the point (-1, 5)
ex. Find the distance between the lines $3x - 4y = 1$ and $3x - 4y = 10$

ex. Find the radius of the circle that is tangent to the line $12x + 5y = 4$ and with center at the point C(2, -1)

ex. Find the equation of the acute angle bisector of the intersection created by the lines

$7x - 24y = 8$ and $3x + 4y = 12$
The equation of a gas line is $2x + y = 2$. A factory located at the point $(6, 7)$ will connect with gas line perpendicularly. Find the equation of the connecting line and the length of the pipe required if the units are in miles.

A cylindrical tank, 6ft in radius, lies on its side parallel to and against the side of a warehouse. A ladder, which leans against the building, passes over and just touches the tank, and has a slope of $-3/4$. Find the equation for the line of the ladder and the length of the ladder.
Functions:

ex. The constant function: \( f(x) = -2 \), find \( f(0) = \), \( f(-4) = \)

what are the x and y-intercepts? ______________

ex. The identity function: \( f(x) = x \), find \( f(0) = \), \( f(-4) = \)

what are the x and y-intercepts

ex. \( f(x) = -2x^2 \) \( \Rightarrow \) find \( f(0) = \), \( f(-4) = \)

what are the x and y-intercepts? ______________

Def. (Increasing). Let \( f(x) \) be a function defined on an interval \( I \) with \( x_1 \) and \( x_2 \) in \( I \). We say \( f \) is increasing if for every \( x_1 < x_2 \), \( f(x_1) < f(x_2) \).

(Decreasing) Let \( f(x) \) be a function defined on an interval \( I \) with \( x_1 \) and \( x_2 \) in \( I \). We say \( f \) is decreasing on \( I \) if for every \( x_1 < x_2 \), \( f(x_1) > f(x_2) \).

Additional work on increasing and non-increasing

Look at the definition of increasing function as defined by your author. The idea will be similar with decreasing.

Def. Let \( f \) be a function defined on some interval \( I \) with \( x_1 \) and \( x_2 \) in the interval and \( x_1 < x_2 \). We say that \( f \) is increasing if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \).

Think about what it means for \( f \) to be non-increasing \( \Rightarrow \) if \( x_1 < x_2 \), then \( f(x_1) \) __________ \( f(x_2) \).

If we follow up the idea with non-decreasing we get \( \Rightarrow \) \( f(x_1) \) ________ \( f(x_2) \) whenever \( x_1 < x_2 \).
Now we are ready to prove #39/64.

Show that a function, say $f$, is both nondecreasing and nonincreasing on some interval $(a,b)$ if and only if the function $f(x)$ is a constant function.

Solution:
what do we mean by if and only if?

Both of the statements together

a) If $f$ is nondecreasing and nonincreasing on some interval $(a, b)$, then $f$ is a constant function.

Pf:
Let $f(x)$ be both nondecreasing and nonincreasing. Show $f$ is a constant function.

b) If $f$ is constant function defined on some interval $(a,b)$, then $f$ is nondecreasing and nonincreasing on the interval.

Pf. Let $f(x)$ be a constant function defined on the interval $(a,b)$. Show that $f$ must be nondecreasing and nonincreasing.
Family of lines:

\[ 2x + 3y = 1, \ 2x + 3y = 2, \ 2x + 3y = 3, \ldots \] these lines all have something in common what is it?

What about the lines \[ x + y = 2, \ x + 2y = 2, \ x + 3y = 2, \ldots \]?

The equation \[ y = mx + 3 \] represents a family of lines with \( m \)_________________________
m is called a parameter. For each value of \( m \), you get a different member of the family.

Other examples:

a) \[ y - 3 = m(x + 2) \]

b) \[ x + 2ay = 4a \]

c) \[ y = 2x + b \]

d) \[ \frac{x}{a} + \frac{y}{2a} = 1 \]

Ax + By + C = 0 and Dx + Ey + F = 0 represent ___________________ while Ax + By + C + k(Dx + Ey + F) = 0 ? ___________________

ex. 2x + 3y = 6 and x + 2y = 2 \( \Rightarrow \) (2x + 3y - 6) + k(x + 2y - 2) = 0 represents?

ex. From the example above, find the value of k so that the member has slope 0

passes through the point (-2, 0)
Additional examples on lines

ex. If \( 3x + 4y = 4 \) is the perpendicular bisector of the segment AB, \( A(2,-1) \) and \( B(-1,-5) \)
Find a point on the circle that is also on the line \( 2x + y = 4 \) and AB being the endpoints of a diameter of the circle

ex. Find the acute angle bisector of the angles created by the intersection of the lines

\[
3x + 4y = 12 \quad \text{and} \quad 5x - 12y = 12
\]
Circles

Def. Let $P(x,y)$ be the set of all points that are equidistant from a fixed point (center). The set of points $P(x,y)$ is called a circle, the fixed distance is called the radius $r$, with $C(h, k)$ representing the center of the circle.

Derive the equation for circle with center at $(2, -4)$ with radius $r = 5$.

Do the same but in general – not a specific circle – all circles.

by definition we get the equation in standard form (center-radius form)

Begin with a circle with center $C(h, k)$ and radius $r$.
Let $P(x,y)$ represent a point (any) that satisfies the definition of the circle.

Find the equation that represents this circle.

A circle with center at the point $C(h,k)$ with radius $r$ ➔

Standard Form(center – radius form)

Notice what happens in different cases of “$r$”.

ex. $(x - 2)^2 + (y - 3)^2 = 25$ ➔ _________________________
ex. $x^2 + (y - 2)^2 = 4$ ➔ _________________________

ex. $x^2 + (y + 3)^2 = 12$ ➔ ___________________________
ex. $(x + 1)^2 + (y - 5)^2 = 0$ ➔ __________________

ex. $x^2 + y^2 = -4$ ➔ __________
ex. $(x + 2)^2 + (y - 1)^2 + 9 = 0$ ➔ __________

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ex. Find the equation with center at (-2, -6) and radius 5.

ex. Find the equation of a circle with center at (-2, 3) and diameter of length 12.

ex. Find the equation of the circle with diameter AB, where A(2, -3) and B(-1, 1) are given.

General Form of a circle: What happens if we take the standard form of a circle and expand it out.

\[(x - h)^2 + (y - k)^2 = r^2 \implies\]

Recall: Complete the square of \[x^2 + 4x - 5 = 0\]
ex. Write in standard form.

a) \( x^2 + y^2 - 2y = 1 \)  

b) \( x^2 + 2x + y^2 - 4y = 2 \)

Does the equation \( 4x^2 - x + 4y^2 + 8y - 40 = 0 \) represent a circle? Why or Why not?

What is the radius of a circle that is represented by the following equation \( x^2 + 8x + y^2 - 12y - 12 = 0 \)?
**Geometric Properties of a circle.**

1) Every tangent line to a circle is ______________ to a segment that represents a radius of the given circle at the point of tangency.

2) We can construct a line passing through the point of tangency perpendicular to the tangent line passing through the center.

3) A perpendicular bisector of any chord will pass through the center of the circle.

4) A triangle inscribed in a circle whose vertices are given can be used to find the center and then the radius of the circle.
5) A circle inscribed in a triangle whose sides are given in terms of lines (intersecting lines) can be used to find the center of the circle and then the radius.

Examples:
Find the equation of the circle that

1) is tangent to the line $4x - 3y = 2$ with center at $(3, -2)$

2) is tangent to the line $2x + y = 1$ at the point $(3, -5)$ and has its center on the x axis.

3) tangent to both axes and passes through $(2, -4)$.
   Assume that circle is in quadrant IV, center at $(h, k)$ and radius $r$, where $r > 0$

4) Circle is tangent to the x-axis with its center at $(4, -2)$
5. circle passes through the points A(0, 0), B(2, 0), and C(-4, 6)

6. Find the center and the radius of the circle represented by the following equations. If any special condition applies to the “circle” then state so.

   a) see problems on page 88

   b) see problems on page 88

   c) see problems on page 88
Family of Circles:

A Family of Circles that pass through the intersection of two given circles:

Given \( x^2 + y^2 + D_1x + E_1y + F_1 = 0 \) and \( x^2 + y^2 + D_2x + E_2y + F_2 = 0 \)

we get

\[
( x^2 + y^2 + D_1x + E_1y + F_1 ) + k ( x^2 + y^2 + D_2x + E_2y + F_2 ) = 0
\]

with the restriction that \( k \neq -1 \). Why would we need this restriction?

What happens when \( k \) does equal to \(-1\)?

ex. \(( x - 1)^2 + ( y + 1 )^2 = 4 \) and \( ( x + 2)^2 + ( y + 3 )^2 = 9 \)

Find a family of circles that pass through the intersection of the circles above.

Find the member that passes through the origin.

What do we get when \( k = -1 \)? This is called ________________________________

Circles:

**Standard form:** \(( x - h)^2 + ( y - k)^2 = r^2 \): circle of radius \( r \) with center at \((h,k)\)

**General form:** \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \), where \( A \) and \( C \) are equal. \( \Rightarrow \)

we usually divide by \( A=C \) to leave in the form \( x^2 + y^2 + Dx + Ey + F = 0 \)
Family of circles

Given the equation of two circles $C_1$ and $C_2$, $C_1 + kC_2 = 0$ represents a family of circles that pass through the intersection of $C_1$ and $C_2$ with the restriction that $k$ does not equal $-1$.

ex. $x^2 + y^2 - 2x - 4y - 12 = 0$ and $x^2 + y^2 + 4x - 8y - 16 = 0$

If these circles intersect, then $(x^2 + y^2 - 2x - 4y - 12) + k(x^2 + y^2 + 4x - 8y - 16) = 0$ with $k \neq -1$ represents a family of circles that pass through the intersection of the given circles.

Radical Axis:
If $k = -1$, we get a line $-6x + 4y + 4 = 0$ or $3x - 2y - 2 = 0$. This line is called the radical axis of the given family of circles.

properties of the radical axis
   a) it passes through the points of intersection, if such points exist
   b) it is perpendicular to the segment connecting the radii of the given circles
   c) if the given circles do not intersect, the line may lie somewhere between the two circles.

NOTE: we can talk about a radical axis even when the idea does not seem to exist – at least not as discussed above
What happens when you take the following circles? See if you find the radical axis and explain the relationship between the circles and the axis.

a) $x^2 + (y-2)^2 = 9$
   $x^2 + (y+2)^2 = 1$

b) $x^2 + (y-2)^2 = 9$
   $x^2 + (y+4)^2 = 1$

c) $x^2 + (y-2)^2 = 9$
   $x^2 + y^2 = 1$
**Transformation of coordinates**

a) rotation of axes - save till later

b) translation of axes to a new origin -- Translation of origin - we will look at now

Why ? – To make the connection more between the algebra and the geometry clearer ?

**Translation of origin**

Given a point \( P(x, y) \) on the plane (the rectangular coordinate system) what would the coordinates be labeled if we looked at the point from a different origin.

Construct a new set of axes (the \( x' \) and \( y' \) axes) with origin \( O' \) at (\( h, k \)). From this point what are the coordinates of \( P \).

Let \( P(x, y) \) represent the original coordinates of a given point with respect to the original \( xy \) axes. If the origin is moved to the point \( O' (h, k) \), then \( P \) has new coordinates with respect to this origin

\[
x' = x + h \quad \text{and} \quad y' = y + k,
\]

where \( (h, k) \) represent the new origin, \( (x', y') \) represents the new coordinates of \( P \) with respect to the \( x' y' \) axes.
Examples:

1. Point P(3, -4) is given find the new coordinates if the origin is moved to the point (-2, 5).

2. Find the new equation when the origin is moved to the point (3, -1).
   a) \( x + 2y = 1 \) ➔ ______________________
   b) \( y = x^2 - 6x + 8 \) ➔ ______________________

3. To what point must the origin be moved so that the resulting equation in terms of \( x' \) and \( y' \) will be
   a) free of all first degree terms: \( x^2 + 2x + 3 + y^2 - 4x + 1 \).
   b) free of all constants and one of the first degree terms: \( x^2 + 2x - 4y + 9 = 0 \)
   c) \( x^2 + 4x - y^2 + 2y = 7 \) (free of all 1st degree terms – you can’t make the constant disappear)
More on Functions:

1) Does the equation of a circle ever represent a function?

2) Does the equation of a circle ever represent an even or odd “relation” (why are we not using the word function?)

3) What is the domain of each of the following
   a) \( x^2 + (y - 2)^2 = 9 \)
   b) \( (x - 2)^2 + (y + 3)^2 = 4 \)

4) What is the range of each of the following
   a) \( x^2 + (y - 2)^2 = 9 \)
   b) \( (x - 2)^2 + (y + 3)^2 = 4 \)
1. Identify as a line, a parabola, a circle, a point, no graph
   a) \( x^2 = y \) b) \( y - x = 2 \)
   c) \( x^2 + (y - 2)^2 + 9 = 0 \) d) \( x^2 + y^2 = 0 \)

2. Identify the following curve by giving a complete description of the curve.
   \( (x + 2)^2 + (y - 3)^2 = 16 \)

3. Complete the square and write the resulting equation in standard form (center-radius) form.
   \( x^2 + y^2 - 8y - 4 = 0 \)

4. Describe the family of lines represented by :
   a) \( x + 2y = C \) \( \Rightarrow \) __________________________
   b) \( \frac{x}{2} + \frac{y}{b} = 2 \) \( \Rightarrow \) __________________________
   c) \( (x + 2y + 3) + k(2x - 3y + 5) = 0 \) \( \Rightarrow \) __________________________

5. Find the directed distance from the line \( 3x - 4y = 1 \) to the point \( (4, -1) \) \( \Rightarrow \) ______________

6. What is the distance between the lines \( 4x + 3y = 4 \) and \( 4x + 3y = 0 \) \( \Rightarrow \) ________________
1. Factor
   a)  $x^2 - 25 = \underline{\hspace{2cm}}$
   b)  $x^2 + 4x - 5 = \underline{\hspace{2cm}}$

2. True or False
   a. Equilateral triangles have all angles of measure $60^\circ$  
      \underline{\hspace{2cm}}
   b. Congruent triangles are always similar
      \underline{\hspace{2cm}}
   c. $|c^2| = (c)^2$ \underline{\hspace{2cm}}

3. Find each of the following absolute values.
   a)  $|3 - 12| = \underline{\hspace{2cm}}$
   b)  $|-2\pi| = \underline{\hspace{2cm}}$
   c)  $|x| = \underline{\hspace{2cm}}$ if $x$ represents a whole number.

4. Find the distance between $A(4, 1)$ and $B(0, -2)$. \underline{\hspace{2cm}}

5. Use distances to determine if the following points are co-linear.
   A(3, 3), B(0, 1), C(9, 7)
1. Factor

   a) $x^2 - 3x - 4 = \underline{\hspace{2cm}}$
   b) $3x^2 - x = \underline{\hspace{2cm}}$

   answer: $(x - 4)(x + 1)$
   answer: $x(3x - 1)$

2. Find the determinant

   $\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$

   ans.: $15 - 8 = 7$

3. Find $180^\circ$ in radians

   $180^\circ = \underline{\hspace{2cm}}$ radians (exact)

   answer: $\pi$

4. Use the given numbers on the right triangle to find the missing side.

5. Find the slope of a line $L$ with angle of inclination $150^\circ$. (exact answer)

   $-1/\sqrt{3}$ this comes from the $30^\circ - 60^\circ - 90^\circ$ right triangle

6. Find the angle of inclination of a line with slope $\sqrt{3}$.

   $\phi = (\tan^{-1}\sqrt{3}) = 60^\circ$

7. Two lines are parallel. The slope of one is $r$, what is the slope of the second line. ______

   Two lines are perpendicular. The slope of one is $r$, what is the slope of the second line. ______

   answer: part a) $r$ – parallel lines have equal slopes, part b) $-1/r$ - perpendicular lines have negative reciprocal slopes.
1. Which of these relations are just RELATIONS and which can be classified as FUNCTIONS?
   a) \( y = 2x \)  
   b) \( y = \frac{3}{x - 4} \)
   c) \( y^2 = 4x \)

2. Give an analytic proof of the following statement.
   The diagonals of a square are of equal length.

3. Find the point \( P \) that is half-way between the points \( A(4, -1) \) and \( B(-2, -7) \) \( \Rightarrow \) _________

   What about the point \( P(x, y) \) that is 1/3 of the way from \( A \) to \( B \) ? \( \Rightarrow \) _________
1. What is the slope of the vertical line passing through the point P (-2, 4)? __________

2. Which of these represents a line? Which one represents a parabola? Which one represents neither?

\[ x = 1 - 2y \quad x^2 = 1 - y \quad y = |x| \quad y^2 = x^2 - 1 \]

______________  _____________  _________  ______________

3. Which of the above four is not a function? ________________

4. Find the slope and the y-intercept of the curve \( y = 2x - 5 \). \( m = \) ________  y-int = ______

5. Prove or disprove.

The diagonals of a rhombus are perpendicular.
1. Sketch the graph of 
   \[ y = 2x + 4 \]

2. Find the equation of the line that is perpendicular to \( 2x - y = 4 \) and passes through the origin.

3. Find the slope of the line \( 2x - y = 4 \)

4. Find the x and y-intercepts of the following relation \( 2x - y^2 = 4 \)
   \[ x \text{-int: } \quad \text{y-int: } \]

5. An isosceles right triangle has how many right angles? 

6. Find the solution of the following system of equations –
   \[
   \begin{align*}
   2x - y &= 5 \\
   3x + 2y &= 4 
   \end{align*}
   \]