Chapter 3. Conics

The general quadratic equation in two variables: \[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]
can represent either a nondegenerate conic or a degenerate conic.

**Nondegenerate Conic:** circle, ellipse, parabola, hyperbola

**Degenerate Conic:** a point, a single line, two intersecting lines, two parallel lines, no graph

Case II. If \( B \neq 0 \), then it is hard to determine what situation we have – we will look at this later
this is the case in which most of the degenerate cases will appear.

**Case I: Assume \( B = 0 \).**
Consider equations (relations) of the form \[ Ax^2 + Cy^2 + Dx + Ey + F = 0 \]

**Circle:** if we have \( A = C \) and there is a positive radius – you can not see this unless you complete the square

\[
\begin{align*}
x^2 + (y-2)^2 &= 4 \\
x^2 + y^2 + 12 &= 0 ? \\
(x-1)^2 &= -y^2 ?
\end{align*}
\]

\[
\begin{align*}
x^2 + (y+3)^2 - 2 &= 0 \\
x^2 + y^2 + 2x - y &= 4 \\
2x^2 + 2y^2 - 1 &= 0
\end{align*}
\]

**Ellipse:** \( A \neq C \) and \( A \cdot C > 0 \), i.e. \( A \) and \( C \) have the same sign – assume there is a positive “radius”

(there is an \( x^2 \) and a \( y^2 \))

\[
\begin{align*}
2x^2 + x + 4y^2 - y - 12 &= 0 \\
5x^2 + 2y^2 - x - 2y &= 4 \\
x^2 &= -4y^2 ?
\end{align*}
\]

**Parabola:** Either \( A = 0 \) or \( C = 0 \) (only one of them) – there is only one quadratic variable and the second variable is linear \( (D \neq 0 \) or \( E \neq 0 \) – whichever represents the variable with no quadratic term)

\[
\begin{align*}
2x + y^2 - y + 3 &= 0 \\
x^2 - 2x - y &= 1 \\
x = 1 - 2y - y^2
\end{align*}
\]

**Hyperbola:** \( A \cdot B < 0 \), \( A \) and \( B \) have opposite signs (“radius” \( \neq 0 \))

\[
\begin{align*}
y^2 - 2x - 4x^2 + y &= 12 \\
2x^2 - 3y^2 - x - 3y - 1 &= 0
\end{align*}
\]
Non-degenerate Conics:
Parabola, Ellipse, hyperbola

**Parabola**
Def. The set of points \( P(x,y) \) that are equidistant from a line and a fixed point is called a parabola. We call the fixed line the directrix of the parabola, and the fixed point is called the focus.

Begin: Let the fixed point be represented by \( F(a, 0) \), \( a > 0 \) and let \( x=-a \) be the fixed line.

Graph: Construct: the graph and the equation that represents it

We are selecting a very special case. If we vary a little we can generate a second case of a parabola but by no means all possible equations.

What if the fixed point is at \( F(a, 0) \) and \( a < 0 \) with fixed line \( x=-a \)? Is the graph changed? How about the equation?

Consider two other possible cases

3) Fixed point at \( F(0, a) \), \( a > 0 \) and fixed line \( y = -a \). What changes have occurred?

4) Fixed point at \( F(0, a) \), \( a < 0 \) and fixed line \( y = -a \). What changes?

Conclude our constructions on this page with the following property.
**Property.**

If we are given a parabola with focus at the point \( F(a, 0) \) and directrix at \( x = -a \) ("midpoint of F and \( x = -a \)" at origin), where \( a > 0 \), then the equation of this parabola is given by

\[
y^2 = 4ax \quad \text{and it opens to the right (} a > 0 \)
\]

Similarly, if \( a < 0 \), then it opens to the left

Graphs for parabola with directrix \( x = -a \):

\[
\begin{align*}
a > 0 \, , \, \text{focus } F(a, 0) & \quad a < 0 \, , \, \text{focus } F(a, 0) \\
\end{align*}
\]

-ex. \( y^2 = -12x \) → opens: _______________ \( 4a = _______ \) so \( a = _______ \) focus: _______

-ex. \( y^2 = 24x \) → opens: _______________ \( 4a = _______ \) so \( a = _______ \) focus: _______

-ex. \( y = -x \) → opens: _______________ \( 4a = _______ \) so \( a = _______ \) focus: _______
Property:
If we are given a parabola with focus at the point \( F(0,a) \) and directrix at \( y = -a \), where \( a > 0 \), then the equation of this parabola is given by
\[
x^2 = 4ay
\]
and it opens upward.
If \( a < 0 \), the parabola opens downward.

Construction:
\( a > 0 \), \( F(0, a) \)
\( a < 0 \), focus \( F(0, a) \)

Parts of a parabola: Begin with a concrete example, \( y^2 = 24x \)

\[
\begin{align*}
\text{ex. Find the graph and the domain of each of the following equations.} \\
\text{a) A parabola with vertex at } V(0, 0), \\
\text{axis of symmetry on x-axis and } a = -2 \\
\text{b) A parabola represented by the equations } y^2 = 12x
\end{align*}
\]
ex. Find the graph and the range of each of the following relations.

a) a parabola with directrix at \( x = -2 \) and focus at \( F(0, 2) \)

b) a parabola represented by the equation \( x^2 = -24y \)

ex. Find the x-intercept of \( y^2 = 16x \). Define vertex, length latus rectum (l.r.)

ex. Find the equation of the parabola that

a) has vertex at the origin with focus at \((0, -5)\)

b) has directrix \( x + 3 = 0 \), vertex at the origin

c) passes through the point \((-2, 4)\) with axis of symmetry parallel to the y-axis and vertex at the origin.
What happens when the vertex is not at the origin?

Construction: use the formulas from translation of origin.

Assume that you have a parabola with focus at \( F(h+a, k) \) and directrix at \( x = h - a \)

Recall the translation formulas:
\[
    x = x' + h \quad \text{and} \quad y = y' + k
\]

You can construct the other three in the same manner.

Property.
Given a parabola with vertex at \((h, k)\) and

a) focus at \((h + a, k)\)

the equation is given by

b) focus at \((h, k + a)\)

the equation is given by

Also,

**Parabola with vertex \((h, k)\):**

length of l.r. = \(|4a|\), \(a\) is the distance from the vertex to the focus, \(a\) also represents the distance from directrix to vertex, axis of symmetry passes through VF – vertex, focus
The equation of a parabola that opens up or down is given by

Standard form: \((x - h)^2 = 4a(y - k)\), opens up if \(a > 0\), down if \(a < 0\) ---
or

General form: \(x^2 + Dx + Ey + F = 0\)

The equation of a parabola that opens to the sides is given by

Standard form: \((y - k)^2 = 4a(x - h)\), opens right if \(a > 0\), opens left if \(a < 0\)
or

General form: \(y^2 + Dx + Ey + F = 0\)

Symmetry: We can talk about a curve being symmetric about a line or a point. We can say a curve is symmetric about the x-axis, about the y-axis, or about the origin -- otherwise it is its axis of symmetry is just some line.

x-axis: if \((x, y)\) is a point on the curve, then so is \((x, -y)\)
y-axis: if \((x, y)\) is a point on the curve, then so is \((-x, y)\)
origin: if \((x, y)\) is a point on the curve, then so is \((-x, -y)\)

Note: if you have two symmetries, you have the third one.
More examples:

Find the equation of the parabola that has

a) vertex at (3, 2), focus at (3, 4)

b) vertex at (4, 1) with directrix y = -3

c) focus at (2, 0) with y = -3 as its directrix

d) passing through the points: A(0, 0), B(4, 0) and C(-2, 1)

Write in standard form and graph.

\[ x^2 + 4y + 8 = 0 \]
\[ x^2 - 12x - 16y - 60 = 0 \]

Prove that

1. If a curve is symmetric about the x and y axes, then it must be symmetric about the origin.

2. If a curve is not symmetric about the origin and is symmetric about the y-axis, then it can not be symmetric about the x-axis.
**Ellipse**

**Ellipse:** The set of all points \( P(x,y) \) in a plane such that the sum of the distances from two fixed points \( F \) and \( F' \) is constant is called an ellipse. The fixed points are called the foci of the ellipse.

WLOG let \( 2a \) represent the constant sum (sum of the distances). The midpoint of the segment connecting the foci is called the center of the ellipse.

**Thm.** The equation of the ellipse with foci at \( F(c, 0) \) and \( F'(-c, 0) \) with \( c > 0 \) and constant sum = 2a (sum of distances) is given by

\[
\frac{(x)^2}{a^2} + \frac{(y)^2}{b^2} = 1
\]

If the foci are at \( F(0, c) \) and \( F'(0, -c) \) with \( c > 0 \) and constant sum = 2a, then the equation is given by

\[
\frac{(x)^2}{b^2} + \frac{(y)^2}{a^2} = 1
\]

**Construction (outline of proof)**

You have an ellipse with foci at \( F(c, 0) \) and \( F'(-c, 0) \), \( c > 0 \). Assume that the sum of the distances is equal to 2a, \( a > 0 \).

Let \( d_1 \) = distance between \( F \) and \( P = d_2 \) = distance between \( F' \) and \( P = \)

We know that \( d_1 + d_2 = 2a \). \( \rightarrow \sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + (y-0)^2} = 2a \)

Notice: if 2a = length of major axis, 2b = length of minor axis, 2c = distance between the foci,
then what is the relationship between a, b, and c?

In a similar way, we get the equation of an ellipse with foci at $F(0, c)$ and $F'(0, -c)$, $c > 0$.

Use the following equation and graph to study the parts of the ellipse.

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Let $V$ and $V'$ be the x-intercepts, $B$ and $B'$ be the y-intercepts, $F$ and $F'$ the foci; keep in mind that $b^2 = a^2 - c^2$.

The segment $VV'$ forms the major axis – the length of the major axis is $2a$.

The segment $BB'$ forms the minor axis – the length of the minor axis is $2b$.

Midpoint of $VV'$, $BB'$, or $FF'$ is called the center of the ellipse.

Parts of an ellipse:

**Let's use the example above to illustrate the parts of an ellipse**

<table>
<thead>
<tr>
<th>center,</th>
<th>major axis,</th>
<th>minor axis:</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertices:</td>
<td>endpoints of minor axis:</td>
<td>latus rectum(l.r.):</td>
</tr>
</tbody>
</table>
ex. Find the equation of the ellipse that

1) has foci at $F'(\ -4,\ 0)$ and $F(4,\ 0)$, endpoint of minor axis at $B(0,\ 2)$

2) Find the equation of the ellipse with minor axis of length 10, vertices at $V(6,\ 0)$ and $V'(-6,\ 0)$

What if the center of the ellipse is not at the origin? As we did with the parabola, we move the origin to the center of the ellipse to get a simple equation and then replace $x'$ and $y'$ to get our original variables.

Consider the ellipse with center at $C(h,\ k)$ and focus at $F(\ h+c,\ k)$ and $F'(\ h-c,\ k)$ (horizontal ellipse)

With a little bit of work we end up with:

Construct a new set of axis (new origin) at $(h,k)$

Begin with the standard equation of an ellipse with center at $C(h,k)$ → 

$$\left(\frac{x'}{a^2}\right)^2 + \left(\frac{y'}{b^2}\right)^2 = 1$$

Use $x' = x + h$ and $y' = y + k$ (or $x' = x - h$ and $y' = y - k$) to obtain an equation in terms of $x$ and $y$.

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Similarly, if the center is at $C(h,\ k)$ and focus is at $F(h,\ k + c)$ and $F'(h,\ k - c)$ (vertical ellipse), we get the following equation.

Begin with the standard equation of an ellipse with center at $C(h,k)$ → 

$$\left(\frac{x'}{b^2}\right)^2 + \left(\frac{y'}{a^2}\right)^2 = 1$$

Use $x' = x + h$ and $y' = y + k$ (or $x' = x - h$ and $y' = y - k$) to obtain
How do you know which case you have?  ans.  $a^2$ is always?  _________________

**Forms of an ellipse:**

**Standard Form:**

1) The two equations above (previous page)

2) The two equations above (previous page)

**General Form:**

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$ with the condition that $A \cdot C > 0$

**Examples:**

ex. Find the equation of the ellipse with vertex V(4, -1), focus F (4, -5) and center C(4, -6)

ex. Find the equation of the ellipse with major axis horizontal and of length 20, ends of minor axis at M(3, 2) and M(−2, 2)
Eccentricity
An ellipse also has eccentricity (roundness) defined by $e = \frac{c}{a}$

- $e$: close to 1 $\rightarrow e$ is close to 1 $\rightarrow \frac{c}{a}$ is close to 1. Use $b^2 = a^2 - c^2$.

- $e$: close to zero $\rightarrow e$ is close to 0 $\rightarrow \frac{c}{a}$ is close to 0. Use $b^2 = a^2 - c^2$.

ex. Find the equation of the ellipse with center at $C(0, 4)$, vertex at $V(12, 4)$ and $e = \frac{5}{6}$

Write the following equation of an ellipse in standard form: $x^2 + y^2 + 2x - 4y - 20 = 0$
Like the circle; you have to make sure that the “radius” of the ellipse is positive.
HYPERBOLA

Def. The set of all points P(x,y) such that the difference of the distances from two fixed points (foci) to P(x,y) is constant (2a) is called a hyperbola.

Thm. A hyperbola with foci at \( F(-c, 0) \) and \( F(c, 0) \) with fixed distance 2a (difference of distances) is represented by the equation

\[
1) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

If the foci is at \( F(0, c) \) and \( F'(0, -c) \) then the equation is

\[
2) \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1
\]

Construction (Proof) of 1). The second one would be similar.

Begin with a hyperbola with foci at \( F(c,0) \) and \( F'(-c,0) \), \( c > 0 \). Let \( P(x,y) \) be the set of all points such that the difference of the distances from the fixed points to \( P(x,y) \) is constant, say 2a.

Let \( d_1 = \text{distance between } F \text{ and } P \), \( d_2 = \text{distance between } F' \text{ and } P \)

We know that \( d_1 - d_2 = \pm 2a \) (one or the other, depending on the size of \( d_1 \) and \( d_2 \); which is bigger)

We know that \( d_1 - d_2 = \pm 2a \). \( \rightarrow \sqrt{(x - c)^2 + (y - 0)^2} + \sqrt{(x + c)^2 + (y - 0)^2} = \pm 2a \)

Square like crazy – and use your algebra to end up with \( \rightarrow \)

The midpoint of \( FF' \) is called the center.

Case 2) would be done in a very similar way.
How do you know which case we have? Which is a and which is b?

Let’s use an actual example to describe the different parts of a hyperbola.

\[ \frac{x^2}{16} - \frac{y^2}{9} = 1 \]

We can find the x-intercepts but there will not be any y-intercepts.

Let \( V \) and \( V' \) represent the x-intercepts. The segment \( VV' \) forms the major – transverse axis; length is 2a

There is no y-intercept but there is a minor axis – conjugate axis; \( \) Let \( B \) and \( B' \) represent the endpoints of the minor axis; length is 2b

The midpoint of the segments \( FF' \), \( VV' \), or \( BB' \) is the center of the ellipse.

Just like the ellipse we have a relationship between \( a^2 \), \( b^2 \), and \( c^2 \rightarrow \) ________________

Why?

Again, what if the center is away from the origin at say point \( P(h,k) \)

We get either

\[ 1) \ \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \]

\[ 2) \ \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1 \]

Notes:

1. With an ellipse \( b^2 = a^2 - c^2 \), what’s is the relationship with a hyperbola? \( b^2 = \) ____________

2. the general form of a hyperbola; \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \), \( A \cdot C < 0 \)
ex. Find the equation of the hyperbola that

a) has focus at \( F' \ (4, -2) \) and \( F (4, 2) \)

b) transverse axis of length 12, conjugate axis of length 16, foci at the points _______________

c) What are the asymptotes of the hyperbola --- represented by the equation

\[
\frac{(x-2)^2}{9} - \frac{(y+2)^2}{16} = 1
\]

ex. given \( B \) and \( B' \) and \( F \) find the equation : \( B(2, -3) \), \( B'(-2, -3) \), \( F(0, 1) \)
Problems to study.

1. (Whispering Gallery): Any sound from one focus of an ellipse reflects off the ellipse directly back to the other focus. Suppose a whispering gallery has the shape of a semi-ellipse. The distance between the foci is 24 m while the height of the gallery at its center is 5m. Find the distance that sound travels as it leaves one focus and travels to the other.

2) A field is in the shape of an ellipse with major axis of length 185 meters and minor axis of length 155 meters. A track 1 meter wide surrounds this field. Find the area of the track.
   Note: an ellipse with axis of length 2a and 2b has area $A = ab\pi$

3) An exhibition tent is in the form of a cylindrical surface with each cross-section a semi-ellipse having base 20 feet and height 8 feet. How close to either side of the tent can a person 5 feet tall stand straight up?

4) A TV satellite antenna consists of a parabolic dish with the receiver placed at its focus. The dish can be described by rotating the parabola $y = \frac{1}{12}x^2$ about its axis of symmetry, where $-6 \leq x \leq 6$ and $x$ is measured in feet. How deep is the dish, and where should the receiver be placed in relation to the bottom of the dish.

5) The cable of a suspension bridge is in the shape of a parabola. The cable extends over 200 feet of highway. The longest supporting wire on either end of the cable is 100 feet. The shortest supporting wire which is in the middle is 30 feet. Find the length of the supporting wire that is 40 feet from the longest supporting wire.

6) Will be filled in later: hyperbola -
Additional Review over non-degenerate conics

1. Identify each of the following as an ellipse, a hyperbola, a parabola, a circle, a line, none of the above
   a) \(3x^2 - 4x + 5y^2 + 2y = 100\)
   b) \(x^2 + 2x - 3y - 100 = 0\)
   c) \(4x^2 + x + 4y^2 - 2y - 20 = 0\)
   d) \((2x - 1) + (3x + 2) = 16\)
   e) \(2x^2 - x - y - y^2 = 4\)

2. Write in the corresponding standard form.
   a) \(x^2 + 2x + 2y^2 - 8y - 4 = 0\)
   b) \(2x^2 - 4x - 2y + 12 = 0\)
   c) \(x^2 - 2x - 2y^2 - 4y - 12 = 0\)
Give a rough but accurate sketch of all three types of curves.

a) \( x^2 - 4x + 2y - 2 = 0 \)

b) \( 9(x + 2)^2 + 16(y-1)^2 = 25 \)

c) \( (x - 2)^2 - 25(y + 3)^2 = 25 \)

True or False: \( b^2 = a^2 - c^2 \)

How do you find the asymptotes of a hyperbola?
Brief Review over Conics

General Quadratic Equation in two variables: \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \)

Degenerate Conics: line, two parallel lines, two intersecting lines, one point, no graph

Nondegenerate Conics: circle, ellipse, parabola, hyperbola

Easier to recognize if \( B = 0 \). Assume that \( B \) is equal to zero. The general quadratic equation in two variables is represented by
\[ Ax^2 + Cy^2 + Dx + Ey + F = 0 \]

Circle: If \( A = C \) (and there is a positive radius)
- Standard Form:
- General Form:

Properties:

Ellipse: If \( A \cdot C > 0 \) (same sign and there is a positive radius)
- Standard forms:
- General Forms:

Properties:

\[ b^2 = a^2 - c^2 , \text{ length of l. r.} = 2b^2/a , \text{ endpoints of major axis are called vertices, } a > b, \ a > c \]
Parabola: IF \( A \cdot C = 0 \) (either \( A \) is zero or \( C \) is zero, but not both of them)

Standard Forms:

General Forms:

Properties: length of l.r. = 4a,

Hyperbola: \( A \cdot C < 0 \) (opposite signs)

Standard Form:

General Forms:

Properties: \( b^2 = c^2 - a^2 \), \( c > a \), \( c > b \), length of l.r. is same as ellipse
Algebraic Curves

Graphs of polynomial curves $y = f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = P(x)$

or quotient of polynomials: $g(x) = \frac{b_mx^m + b_{m-1}x^{m-1} + \ldots + b_1x + b_0}{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0} = \frac{Q(x)}{P(x)}$

With the use of a graphing calculator we can sketch a graph of some of these type of curves

Without a calculator we have to make use of some of the properties:

- x intercepts
- y intercepts
- vertical asymptotes
- horizontal asymptotes
- slant asymptotes

ex. graph $y = x^2 - 4x - 1$

ex. $f(x) = x^3 + x$
ex. \(g(x) = x(x-1)(x+2)\)

More Algebraic Curves:
Vertical, Horizontal, and Slant Asymptotes

Recall Parabolas: \(f(x) = ax^2 + bx + c\) ---

What about hyperbolas: \(y^2 - x^2 = 4\).

Recall Domain:

What is the domain of

\[
y = \frac{x}{x-2}
\]

\[
f(x) = \frac{3}{x+1}
\]

Domain:__________________       Domain: ____________________
These last two examples were **algebraic curves**.

**Suppose we have a curve in the form** \( f(x) = P(x) / Q(x) \), **where both** \( P(x) \) **and** \( Q(x) \) **are polynomial functions.**

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad \text{and} \quad Q(x) = b_m x^m + \ldots + b_0
\]

**Vertical Asymptotes:** we can find vertical asymptotes by setting \( Q(x) = 0 \)

**Horizontal Asymptotes:**

Suppose \( P(x) \) is of degree \( n \) (degree of numerator), and \( Q(x) \) is of degree \( m \) (degree of denominator)

1) if \( n > m \), there is no horizontal asymptote

\[
f(x) = \frac{x^2 + 3x - 2}{x + 2}
\]

2) if \( n < m \), the horizontal asymptote is represented by \( y = 0 \), the x-axis.

\[
f(x) = \frac{x^2 + 3x - 2}{x + 2}
\]

3) if \( m = n \), there is a horizontal asymptote, represented by \( y = b/a \), where \( b \) and \( a \) are the leading coefficients

\[
g(x) = \frac{4x^4 + 3x - 4}{5x^4 + x^2 + 2} \rightarrow \text{horizontal asymptote} \rightarrow y = 4/5
\]

The curve represented by \( y = \frac{e^x + e^{-x}}{2} \) has asymptotes – but linear asymptotes. **We will study these in more detail later.**
ex. Find the vertical asymptotes of

a) \( f(x) = \frac{x - 1}{x + 2} \)

Vertical: 
Horizontal:

\[ \text{Graph} \]

b) \( g(x) = \frac{3}{x - 3} \)

Vertical: 
Horizontal:

\[ \text{Graph} \]
c) \[ h(x) = \frac{x}{x^2 - 16} \]

Vertical: 
Horizontal: 

\[ \text{Graph of } h(x) \]

---

d) \[ f(x) = \frac{x - 4}{8} \]

Vertical: 
Horizontal: 

\[ \text{Graph of } f(x) \]

---

e) \[ f(x) = \frac{x^2 - 4x}{x^2 - 16} \]

Vertical: 
Horizontal: 

\[ \text{Graph of } f(x) \]
f) \( g(x) = \frac{x^2 - 1}{x + 1} \)

Vertical:  
Horizontal:

Note: a curve does not usually cross a vertical asymptote but it can cross a horizontal asymptote.
Slant Asymptotes:
If \( n = m + 1 \) (the degree on the numerator exceeds the denominator by), we do not have a H.A. but we usually have a slant asymptote. We can find it by dividing the denominator into the numerator (long hand division or by using synthetic division).

ex. Find the slant asymptote of

\[
h(x) = \frac{x^2 - 4x - 12}{x + 2}
\]

→ it seems like if there is a slant asymptote, but it does not. Why?

ex. g) \( y = \frac{x^2 + 2x + 3}{x + 2} = \frac{(x^2 + 2x)}{(x + 2)} + \frac{3}{x + 2} = \frac{x(x+2)}{(x+2)} + \frac{3}{x+2} = x + \frac{3}{x+2}
\)

→ \( y = x + \frac{3}{x+2} \), \( y = x \) represents the slant asymptote.

Graph it.
Sketch the graph of each of the following algebraic curves.

1) \[ y = \frac{3x(x + 2)}{x - 3} \]

2) \[ y = \frac{x^3 - 9x}{x + 2} \]

3) \[ g(x) = \frac{2x^2 + x}{x^2 + x} \rightarrow y = \frac{2x + 1}{x + 1} \text{ with } x \neq 0. \text{ Why?} \]

There is a horizontal asymptote: \( y = 2 \). Why?

Does the curve \( g(x) \) cross \( y = 2 \)? Why?
Take a look at the second example again. \[ y = \frac{x^2 - 9}{x + 2} \]

\[ y = \frac{x(x-9)}{(x+2)} \]  \rightarrow To remind you about synthetic division divide

\[
\begin{array}{c|ccc}
-2 & 1 & 0 & -9 \\
 & & -2 & 4 \\
\hline
 & 1 & -2 & 5 \\
\end{array}
\]

The quotient: \( x - 2 \)
The remainder: 5

If \( y = \frac{x(x-9)}{(x+2)} \), then \( y = x - 2 + \frac{5}{x + 2} \)  \rightarrow S.A.  \rightarrow y = x - 2

The one question we have not asked: does it cross the asymptote(slant):