Transcendental Functions

Trig. functions –

Graphs:
\[ y = \sin x \quad y = \cos x \]
\[ y = \tan x \quad y = \sec x \]
\[ y = \cot x \quad y = \csc x \]

\[ f(x) = a \sin(bx + c) \]
Sine curve with amplitude = __________, period = _______ and shift = __________

Similar for \[ f(x) = a \cos (bx + c) \]

Similar for \[ f(x) = a \tan(bx + c) \]
Problem#28 page 189
A jet aircraft on landing makes a sound with the equation $150 \sin 1000\pi t$, measured in decibels while $t$ is measured in seconds. What is the frequency of the sound? What is the amplitude?

Which of the six basic trig functions is even and which is odd?

odd: _________________________________        even:________________________________

Recall:

Identities:
\[
\cos^2 x + \sin^2 x = 1 \\
1 - 2\sin^2 x = 1 \\
2\cos^2 x - 1 = 1 \\
\cos^2 x - \sin^2 x = 1
\]

A point on the unit circle with coordinates $P(x,y)$ can be written in terms of the cosine and sine function.
Exponential and Logarithmic Functions

\[ f(x) = 2^x \] and in general \[ g(x) = a^x, \ a > 0, \ x \text{ is a real number} \]

Table Method:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

y-axis

x-axis

Domain: ________________  Range: ________________
x-int: ______  y-intercept: ______ Are there any asymptotes? ________________

Variations of this type of exponential functions.

\[ f(x) = 2^x \]
\[ f_2(x) = -2^x \]
\[ f_3(x) = -2^{-x} \]

Examples:

1. #14 and #15/194

\[ A = Pe^{0.1000t} \]: A-amount present after \( t \) years if \( P \) dollars are invested at \( r \) percent -- If you invest $1000 at 10% per year compounded continuously, how much will be in the account after 2 years?

2. \( f(t) = \begin{cases} 
    t^2 & \text{if } 0 \leq t \leq 2 \\
    t + 2 & \text{if } 2 < t \leq 3 \\
    100.5e^t & \text{if } 3 < t 
\end{cases} \) How many cases were there in the fifth and seventh week?

y: new cases of influenza measured in hundreds, \( r \): time in weeks
Application of exponential functions – Hyperbolic Functions.

ex. We do not really need any help in graphing $y = 2x + 3$. Think of this function as the sum of two functions.

$y = (2x) + (3) \rightarrow f(x) = 2x$ and $g(x) = 3$.

Graph this individually – and then combine them to get the graph of $y = 2x + 3$

Try the same approach to graph each of the following functions.

$y = \frac{e^x + e^{-x}}{2}$ and $y = \frac{e^x - e^{-x}}{2}$

$y = e^{x/2} + \frac{e^x}{2}$

$f(x) = e^{x/2}$ and $g(x) = \frac{e^x}{2}$

We define

\[ \sinh x = \quad \text{and} \quad \cosh x = \]

\[ \text{hyperbolic sine} \quad \text{hyperbolic cosine} \]
Relationships between trig functions and hyperbolic functions

1. Areas: page 193(text)

2. Identities: similar – but not exact

Graphs of hyperbolic trig. functions. (asymptotes ?)

\[ y = \sinh x \quad y = \cosh x \]

\[ y = \tanh x \]
Logarithms
From algebra or trigonometry you remember logarithms;

We write \( y = \log_b x \) and say “the logarithm of \( x \) base \( b \) is equal to \( y \)” to mean

\( y \) is an exponent (the logarithm is an exponent) so that \( b^y = x \), \( b > 0 \).

ex. \( \log_3 81 = \) _______ \( \log_{10} 100 = 0 \)

ex. What must \( x \) be so that

\( \log_b 32 = -5 \)
\( \log_{1/4} x = 2 \)

Graph of a logarithm. Sketch the graph of \( y = \log_2 x \)

\( f(x) = \log_2 x \) and in general \( g(x) = \log_b x \), \( b > 0 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>-2</th>
<th>( 1/2 )</th>
<th>( 1/4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Domain: ________________ Range: ________________ x-int: ________ y-intercept: ________

Properties:
1. \( \log(xy) = \log x + \log y \)
2. \( \log(x/y) = \log x - \log y \)
3. \( \log x^y = y \log x \)

Special bases:
common logarithm: if \( b = 10 \) → we write
natural logarithm: if \( b = e \) (irrational #) → we write
Other examples:

#49/page 201

\[ \log x + \log (x +1) = \]

**Inverses: not the same as reciprocals**

We say that functions \( f(x) \) and \( g(x) \) are inverses if \( f(g(x)) = g(f(x)) = x \) for every \( x \) in the domain of \( f \) and \( g \).

ex. \( f(x) = 2x - 1 \)

ex. \( f(x) = \sin x \)

ex. \( f(x) = 2^x \)

Recall:

\[
\begin{align*}
f(x) &= \frac{e^x - e^{-x}}{2} \\
g(x) &= \frac{e^x + e^{-x}}{2}
\end{align*}
\]

Even, odd, neither? Symmetry?
Polar Coordinates –
Need a ray: polar axis, pole
points on a plane can be described in terms of rectangular coordinates or as polar coordinates

In this setting we describe the points in terms of the distance from the pole, \( r \), and a trigonometric angle, \( \theta \), in the order \(( r, \theta )\).

If \( r > 0 \) we plot the point on the terminal side of \( \theta \), but if \( r < 0 \), then we plot on the mirror image (across the origin) of the terminal side of \( \theta \).

Graph on a polar coordinate system

a) label the polar axis, \( \theta = 90^\circ \), \( \theta = 180^\circ \), and \( \theta = 270^\circ \) and the pole

b) plot the points; \( A(3, 90^\circ) \), \( B(4, 135^\circ) \), \( C(4, -180^\circ) \), \( D(-4, 45^\circ) \), \( E(-2, -45^\circ) \)

Radians <-> Degrees

Remember that \( 2\pi \) radians = \( 360^\circ \) \( \Rightarrow \pi \) radians = \( 180^\circ \) so \( \Rightarrow \pi \) radians / \( 180^\circ \) = \( 1 = 180^\circ / \pi \) radians

Change to radians:

\[ 135^\circ \implies \ ______ \quad \quad \quad \quad \quad -720^\circ \implies \ ______ \quad \quad \quad \quad \quad 2^\circ \implies \ ______ \]
Change to degrees: (radians is implied)

\[
\frac{5\pi}{7} \text{ radians} \implies \text{__________} \quad -6\pi \implies \text{__________} \quad 1 \text{ radian} \implies \text{______}
\]

There is a relationship between polar coordinates and rectangular coordinates. Superimpose the two coordinate systems on some point \(P\).

\[
\text{P}(x, y) \text{ and } P(r, \theta) \text{ represent the same point on a plane. They are related in the following way}
\]

\[
x = r\cos \theta \quad \text{and} \quad y = r\sin \theta
\]

- these are used to change from polar to rectangular coordinates (variables if you are given an equation)

\[
x^2 + y^2 = r^2, \quad \tan \theta = \frac{y}{x} \quad \text{--- these are used to change from rectangular to polar coordinates}
\]

( change from polar variables in an equation to rectangular variables)

Ex. Change to polar coordinates

\[
\left( -4, 4 \right) \implies \text{__________} \quad \left( 0, -4 \right) \implies \text{__________}
\]

\[
\left( 3, 3\sqrt{3} \right) \implies \text{__________}
\]

Ex. Change to rectangular coordinates

\[
\left( 3, 225^\circ \right) \implies \text{__________} \quad \left( -4, \frac{\pi}{3} \right) \implies \text{__________}
\]

\[
\left( 2, -135^\circ \right) \implies \text{__________}
\]
Ex. Write as an equation with polar coordinates (variables)

\[ x + 2y = 4 \]
\[ x = 3 \]
\[ y = -2 \]
\[ 4x^2 + 4y^2 = 16 \]
\[ ax + by = c \]

Ex. Write as an equation with rectangular coordinates

\[ r = -2 \]
\[ r = 2 \cos \theta \]
\[ r = 2 \sin \frac{\pi}{6} \]
\[ r^2 = 2 \sin \theta \]
\[ r = 2 / (1 - \cos \theta) \]
Additional Examples.

Ex.  Change to polar coordinates

\[ x = 4 \Rightarrow \______________________ \quad y = -2 \Rightarrow \______________________ \]

\[ 2x - y = 4 \Rightarrow \______________________ \quad x^2 + y^2 = 16 \Rightarrow \______________________ \]

\[ y^2 = 2x + 2 \Rightarrow \______________________ \quad 4x^2 + y^2 \Rightarrow \______________________ \]

\[ x^2 - y^2 = 1 \Rightarrow \______________________ \]

Ex.  Change to rectangular coordinates

\[ r = -5 \Rightarrow \______________________ \quad \theta = -1 \Rightarrow \______________________ \]

\[ r \sin \theta = -1 \Rightarrow \______________________ \quad r = \sqrt{2} \sin 45^\circ \Rightarrow \______________________ \]

\[ r = \______________________ \Rightarrow \______________________ \]

Graph of some functions in polar coordinates - construct a table of values and plot them just like we do with rectangular coordinates.

ex.  graph  \( r = 2 \sin 3 \theta \).

<table>
<thead>
<tr>
<th>( \theta ) (degrees)</th>
<th>0</th>
<th>10(^o)</th>
<th>15(^o)</th>
<th>20(^o)</th>
<th>30(^o)</th>
<th>....</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta ) (radians)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r = 2 \sin 3 \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Other Special Curves.

Rose Curves:
\[ r = a \sin n \theta \quad r = a \cos n \theta \] with \( n > 1 \) rose curve with either \( n \) loops (if \( n \) is odd) or \( 2n \) loops if \( n \) is even.

ex. see above. \( r = 2 \sin 3 \theta \) \hspace{1cm} \text{ex. } r = 4 \cos 2 \theta

Limacon

Cartesian equation: \( (x^2 + y^2 - 2ax)^2 = b^2 (x^2 + y^2) \)
Polar equation: \( r = b + 2a \cos (\theta) \)

We will use the form: \( r = b + a \sin \theta \quad r = b + a \cos \theta \)

a) if \( b > a \) (absolute value of each), then we have a curve surrounding the origin. See page 220 (not a circle)

ex. \( r = 3 + \sin \theta \quad r = -4 + \sin \theta \quad r = 3 - \cos \theta \quad r = -2 + \cos \theta \quad r = -3 - \sin \theta \)

b) if \( b < a \) (absolute value of each), we have a curve with an inner loop. See page 229

ex. \( r = 2 + 4 \sin \theta \quad r = -2 + 6 \cos \theta \quad r = -2 - 5 \sin \theta \)
Limacon

c) if \( b = a \) (absolute value of each), then we have a **cardioid**. See page 224

ex. \( r = 3 + 3 \sin \theta \) \hspace{1cm} r = -2 + 2 \cos \theta \hspace{1cm} r = -1 + \cos \theta

Lemniscate: \( r^2 = a^2 \sin 2\theta \) or \( r^2 = a^2 \cos 2\theta \)

has two dumbbells either on the \( \theta = 0^\circ \) or on \( \theta = 45^\circ \). See page 230 and 231

ex. \( r^2 = 4 \sin 2\theta \) \hspace{1cm} ex. \( r^2 = 25 \cos 2\theta \)

Concentrate on the equation \( r^2 = 4 \sin 2\theta \)

What would happen if \( \theta \) is chosen in such a way that \( \sin 2\theta < 0 \)?

Using the discussion above explain if \( r^2 = -4 \sin 2\theta \) would have a graph? If so, what would it look like?
1. Write in terms of polar coordinates (variables – r, \( \theta \))
   a) \( x^2 + y^2 = 16 \) \( \Rightarrow \) ____________
   b) \( x = 4y \) \( \Rightarrow \) ____________

2. Write in terms of x and y. (From Polar to Rectangular)
   a) \( r = -2 \) \( \Rightarrow \) ____________
   b) \( x \sin \theta = -2 \) \( \Rightarrow \) ____________

3. Sketch the graph of (Polar Coordinates)
   a) \( \theta = -45^\circ \)
   b) \( r = 4 \sin 45^\circ \) \( \Rightarrow \) ____________
   c) \( r \sin \theta = 2 \)

4. There are three types of limacons; “fat circles”, inner loop, and ________________

5. There are two types of lemniscates; \( r^2 = a^2 \sin 2\theta \) with axis of symmetry is \( \theta = 45^\circ \) and ________________ with axis of symmetry on the polar axis.

6. Describe in some detail (words) as to what the graph of the following curve looks like;
   \( r = 4 \sin 4\theta \)

7. Eliminate the parameter in each of the following equations.
   a) \( x = 2t - 1 \) and \( y = 4t \)
   b) \( x = 2\sin \theta \) and \( y = \cos \theta \)