Abbreviated Notes on Permutations and Combinations

Factorials:

4 \cdot 3 \cdot 2 \cdot 1 = _________  
7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = _________

Def. Let n be a natural number. We define n factorial, written n !,  
\[ n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 \]

Find

\[ 8! = \underline{\phantom{000}} \quad \frac{12!}{10!} = \underline{\phantom{000}} \quad 99! \cdot 100 = \underline{\phantom{000}} \]

Recall: Cross Product Rule

If n(A) = r_1 and n(B) = r_2, then n(A \times B) = \underline{\phantom{000}}

Fundamental Principle of Counting (Multiplication Principle):

If decision 1 can be done in r_1 ways and decision d_2 can be done in r_2 ways, then both decisions can be done together in

r_1 \cdot r_2 

Examples:

1. A class consists of 10 girls and 12 boys. One member from each group will be selected to sing in an upcoming concert. If the pair is selected at random, then how many different pairs are possible?

2. A group of three students is to be selected so that the first student will get $10, the second one will get $5, and the third one will have to pay $15. Assume that the original group consists of 10 students. How many different groups of three are possible? \underline{\phantom{000}}

3. A company classifies employees in terms of the department they work in: 20 work in sales, 15 work in the business office, and 4 are classified as management. An employee can be in only one type of classification. A group of three is to be selected so that each department has exactly one representative. How many different groups are possible?

4. You wake up in the morning and find that you have three available shirts to wear and four pants that will coordinate well with any of the shirts. How many different outfits could be selected? What if you had an additional choice of five pairs of shoes? \underline{\phantom{000}} \underline{\phantom{000}}
Let’s look at these two examples.

A student has to read any three of five books. How many different selections are possible?

A faculty senate consists of 20 representatives. From these twenty a President, Vice President, and a Secretary are to be selected. Assuming that no member may serve in more than one position, how many different ways can the three members be selected (different groups of three)?

**Permutation**

Consider the set \{ a, b, c, d, e, f \}. Find all possible four-letter words – they do not have to make sense.

Notice:
1) order does matter and
2) we will not allow for repetitions

How many are there?

\[ P(n, r) : \text{the number of permutations of a set of } n \text{ objects taken } r \text{ at a time (from above; } n=6 \text{ and } r = 4 \) \]

We also write \( _nP_r \) to represent the same thing.

Formula:
\[
P(n, r) = __________________
\]

Examples:
\[
\text{Find } P(10, 4) = _______________ \quad \text{P}(200, 2) = _______________
\]

Word example:

A graduating class consists of 200 students. At the end of graduation they will exist the auditorium in a random order. The first 20 students will receive a hug and a gift from an anonymous guest. How many different groups are possible if

a) each gift is different? ____________

b) what if all the gifts are the same? ______
Combination: Consider the set \{ a, b, c, d, e, f \}. Find all possible four letter words – they do not have to make sense. Notice in this case we
1) order does not matter and 2) we will not allow for repetitions

How many are there?

Notation:

\[ C(n, r) \text{ or } \binom{n}{r} \text{ all represent the number of combinations of a set with } n \text{ objects taken } r \text{ at a time.} \]

( from example above; \( n = 6 \text{ and } r = 4 \) )

Formula: \( C(n, r) = \)

Examples:

Find \( C(200, 0) = \) \( C(42, 3) = \) \( C(52, 5) = \)

Word Examples:

1. A committee of four is to be selected from a group of 20. How many different committees are possible? _________

2. A deck of cards consist of 52 cards. How many different five card hands are possible? ____________
   How many with exactly three aces? _________
   How many with exactly two diamonds and three spades? ____________
Additional examples of permutations, combinations, and other multiplication principle problems

1. Twenty cars drive through an intersection; 5 are blue cars, 4 are red, and the rest are of a different color. Three are these cars are chosen at random for inspection. What is the probability that the cars selected are
   a) all blue?                                        b) two blue and one red?

2. Twenty children line up for lunch. If they are lined up in a random order, what is the probability that the children are in alphabetical order?

3. Three members of a family are part of a group of 20. A President and VP are to be elected by having slips of papers selected from a hat. Every member fills out their own name on the paper.

   Two slips of papers are selected and held together. The first one selected will be the President.

   What is the probability that two members of the same family will be selected to serve?

4. A starting team will be selected as well as a B team. There are 20 students that sign up; 8 will make the starting team and the remaining players will make up the B team.

   If the coach is a new age coach and will select the starting team by random selection, then what is the probability that the starting team will be made up of the top ten players?

5. A three-digit number is to be created by using the digits {1, 2, 3, 4, 5}. What is the probability that the number created will be greater than 400 if
   a) repetitions are allowed? _________            b) not allowed? _________

6. Consider all five-card hands from a standard deck of cards. What is the probability that
   a) all cards are aces? _________                b) three are aces and two are kings? _________

   c) exactly two are aces? _________              d) at least one is an ace? _________
Other examples:

A die is rolled five times. What is the probability that in these five rolls exactly one is a four? ____________

What is the probability that exactly two of the rolls are fours?

A person drives by a light five times a week. There is a 1:6 odds that he will catch a green light each time he gets to the light.

What is the probability that he catches

a) a green light on all five days? __________

b) on all but one day? _________

c) exactly three days? _________________

d) at least one day? ___________

A tree has a probability of 0.2 of surviving the first year. Ten trees are planted. What is the probability that

Exactly one tree survives? ____________

b) at least one tree survives? ________

In all of these problems;

a) an experiment was performed in which two outcomes were possible; success and a failure

b) experiment was repeated several times; independent trials.

When this occurs we call it a Binomial experiment;

n : represent the number of trials, x : the number of successes

p: success probability q: failure probability

\[ P( X = x ) = \text{probability of having } x \text{ successes in the experiment.} \]

We can make use of the following formula → if the experiment is known to be binomial.

\[ P( X = x ) = \binom{n}{x} p^x q^{n-x}. \]
Let \( X \): represent the possible numerical values of an experiment – a random variable.

ex.
A doctor performs three surgeries per day. Let the r.v. \( X \) represent the number of patients with no insurance.

Assume that the probability distribution of \( X \) is given by

\[
\begin{array}{c|c}
X = x & P( X = x ) \\
\hline
0 & 5/11 \\
1 & 4/11 \\
2 & 1/11 \\
3 & 1/11 \\
\end{array}
\]

Because \( X \) is written in terms of numbers we can now define a mean, variance, as well as a standard deviation.

Ex.
A six-sided die is rolled six times. The faces are labeled as 1, 2, 2, 3, 3, 3.
Let the r.v. \( Y \) represent the value of the outcome.
Write down the values of the r.v. and the probability distribution. Calculate the mean, the variance, and the standard dev.

example:
Let the r.v. \( Z \) have the following probability distribution. Find the standard deviation.
(Select one of three colors; blue, red, white → blue: you lose $2, white: money back, red: win $4)

\[
\begin{array}{c|c}
Z & P ( Z ) \\
\hline
-1 & 5/10 \\
0 & 4/10 \\
4 & 1/10 \\
\end{array}
\]
When the experiment can be classified as a Binomial Experiment – X is called a Binomial r.v.

In this case the mean: \( \mu \) is obtained by the formula \( \mu = np \)

and the variance is obtained by \( \sigma^2 = npq \).

ex1. Suppose that a coin is tossed 124 times and the r.v. \( Y \) represents the number of heads obtained. Find the mean and the standard deviation of \( Y \).

ex2. 40 trees are planted. Each tree has a 15% chance of survival. How many trees do you expect to survive? If \( X \) represents the number of trees that survive, what will be the mean and the standard deviation of \( X \).

ex3. During flu season children are admitted into a hospital with flu symptoms. 30% of those admitted will have the flu. If 100 children were admitted, then find the expected number of children that will have the flu. If \( X \) represents the number of children that have the flu, then find its mean and standard deviation.
Remember Normal Distribution:

When is \( n \) is large enough (say \( n > 26 \)), we can use normal distribution to approximate our solutions to a binomial experiment.

From example 1 on previous page:

Find the probability that you will get exactly 80 heads in the 124 tosses. ________________

Use normal curves to approximate the following
What is the probability that more than 62 heads will come up?

You will get between 60 and 70 heads?

From example 2: What is the probability that less than five trees will survive? ________________

From example 3: What is the probability that between 25 and 35 children will have the flu?