Factorials

The process of multiplying numbers of the form

\[ 1 \cdot 2 \cdot 3 \quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 100 \]

occurs frequent enough that we define this type of multiplication.

**Def.** We define 0 factorial, 0!, by 0! = 1.

Let n be any natural number, say 1, 2, 3, 4, ....

We define n factorial as \( n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1. \)

Look at these examples to better understand the meaning of n!.

\[ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24, \quad 2! = 2 \cdot 1 = 2, \quad 1! = 1 \]

\[ 10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \quad 200! = 200 \cdot 199 \cdot 198 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \]

**Other examples:**

\[ \frac{8!}{10!} = \quad \frac{100!}{98!} = \]

**Notice:**

1) \( 1! = 0! = 1 \)
2) \( n! \) gets big very quickly \( \rightarrow 5! = 120, 6! = 720, 7! = \ldots \)

**More Examples:**

1. \( 7! - 6! = \quad \)
2. \( 4! - 0! = \quad \)
3. \( 5! + 3! = \quad \)
4. \( 20! / 19! = \quad \)

5. \( \frac{4!}{1! + 0!} = \quad \)
6. \( \frac{6!}{3!} = \quad \)

**ex:** \( 6! = \quad \quad 8! / 6! = \quad \quad 4(3! + 2!) = \quad \quad \)
Other Factorial Problems

\[(n!) \cdot (n+1) = \text{_______} \quad \frac{(n-1)!}{(n)!} = \text{__________}\]

\[
\frac{5!}{\text{_______}} = \text{_______} \quad 80! = \text{_______} \quad 300! / 298! = \text{_______}
\]

\[4! - 5! = \text{_______}\]
Introduction

to

Permutations and Combinations

There are problems in mathematics which involve counting, such as how many groups of 5 could you make from a group of 20? That’s easy enough → 4.

But what if we ask the same question in a slightly different form.

A class consists of 20 students.

A group of 5 students is to be selected from the class. If only one group is selected, how many different ways could that group be selected? ABCDE is the same group as ABDEC but not the same as ABDEF.

This is called a combination. The question really is “how many ways can a group of 5 be selected from a group of 20” or even better → “How many distinct combinations of five people can be made from a group of 20 – keeping in mind that on any instance only one group of five is being selected.”

If we label the students as $S = \{ s_1, s_2, s_3, \ldots, s_{20} \}$ we want groups that look like $s_1, s_2, s_3, s_4, s_5$ $s_1, s_2, s_3, s_5, s_4$ $s_1, s_2, s_3, s_4, s_6$, ...., $s_2, s_4, s_6, s_8, s_{10}$, $s_7, s_8, s_1, s_{20}, s_{17}$, and so forth

In the problem above – order did not matter – but what if order did matter.

If we label the students as $S = \{ s_1, s_2, s_3, \ldots, s_{20} \}$ we want groups that look like $s_1, s_2, s_3, s_4, s_5$ $s_1, s_2, s_3, s_5, s_4$ $s_1, s_2, s_3, s_4, s_6$, ...., $s_2, s_4, s_6, s_8, s_{10}$, $s_7, s_8, s_1, s_{20}, s_{17}$, and so forth

keep in mind that order does matter and we have no repetitions - Now how many groups are possible?

ex. Five students are to be selected from a group of 20. The first student selected will get $20, the second $10, the third $5, the fourth gets $1, the fifth gets $0.50. Order obviously matters! Is this the same question as the one above?

This is an example of a permutation. The question remains almost the same – except order matters! “How many ways can a group of five be selected from a group of 20 – if order matters”

or even better “How many distinct permutations of five can be made from a group of 20 – keeping in mind that you are only using one group of five at any given time”

This problem will be answered after we discuss the next topic.
Last time we introduced the idea of factorials and the definitions and formulas for permutations and combinations.

Def. Let \( n \) be a natural number. We define \( n! \) by
\[
 n! = n (n-1)(n-2)\cdots(2)(1)
\]
In the event that \( n = 1 \) or \( n = 0 \) we define \( 1! = 1 \) and \( 0! = 1 \).

Def. A permutation is a collection of \( r \) objects of a given set with \( n \) objects with no repetitions and order matters.

ex1. \( \{ a, e, i, o, u \} \)

Construct all the three letter permutations. that are possible.

ex2. A President and a VP are to be chosen from a group of 10 members. How many different selections are possible if no member can serve more than one position?

Def. A combination is a collection of \( r \) objects of a given set with \( n \) objects with no repetitions and order matters.

ex. 3

Three cards are selected from a standard deck of cards. How many different groups of three cards are possible?

ex4. A department wishes to select a committee of two from a group of 20. How many different committees are possible?

Formulas:
\[
\begin{align*}
\text{P}(n, r) &= \frac{n!}{(n-r)!} \\
\text{C}(n, r) &= \frac{n!}{r!(n-r)!}
\end{align*}
\]

We can use each of these two formulas on the problems above.

1) \quad 2)

3) \quad 4)

Where do these formulas come from?
Think back:

Let \( A = \{ a, b, c, d \} \), \( B = \{ 1, 2, 3 \} \) and \( C = \{ x, y \} \).

You select one symbol from each set and combine all three in the order selected (one from \( A \), one from \( B \), and one from \( C \)).

How many symbols are possible?

We can get this answer from the idea of cross products.

**Multiplication Principle.**

ex5. You got up this morning with a choice of three different pants to wear and five shirts. How many different outfits could you have walked out with?

ex6. You have six pictures to decorated your room with. There are four spots to be filled – the other remaining pictures will be stored away. How many different ways can you decorate your room?

**Multiplication Principle:**

You have decisions \( D_1 \), \( D_2 \), and \( D_3 \) that have to be made and

- decision \( D_1 \) can be made in \( k_1 \) ways,
- decision \( D_2 \) can be made in \( k_2 \) ways,
- decision \( D_3 \) can be made in \( k_3 \) ways

All three decisions can be made in \( k_1 \cdot k_2 \cdot k_3 \) ways.

ex7. A five problem multiple choice quiz is taken. Each problem has five choices – only one of which is correct. How many different answer “sheets” are possible?

What is the probability that you answered exactly one question correctly? __________

ex8. A Pres., a Vice –Pres, and a Sect. are to be chosen from a group of 21. How many different ways are possible for this to happen?
Other examples.

ex.9 Three cards are selected from a standard deck of cards. How many different groups of three cards are possible?

How many groups with three aces?

How many with exactly one ace and two diamonds

What is the probability that the hand consists of

three aces? one ace and two diamonds?

ex10. You take a 10 problem matching quiz in which 15 choices are possible – an answer can only be used once.

How many different ways are there to answer the 10 problems? _________

What is the probability that you will not answer any of the questions correctly? _________

ex11.

A department consists of 20 men and 30 women. 8 of the men have a college degree while 20 of the women have a college degree. A group of 5 is to be chosen. What is the probability that

a) all five are women?

b) at least one is a man?

c) exactly three have a college degree?
Combination: order does not matter (AB = BA) and there are no repetitions
Permutation: order does matter (AB ≠ BA) and there are no repetitions

1. How many ways can a Pres., Vice P be chosen from a group of 12 _________

2. Two new federal judges will be appointed from a group of 20 state judges. How many different selections are possible? If 12 are men and 8 are women, what is the probability that they are both women? ______________

3. How many ways can a committee of 4 be appointed from a group of 8 men and 4 women? ____
   What is the probability that all four are men? __________
   What is the probability that exactly two are women? ________
   What is the probability that at least one is a man? __________

4. How many ways can 6 books be arranged on a shelf? __________

5. How many different “4-letter words” can be made using the blocks ABCD? ______
   How many “two-letter words” if any two of the blocks can be used? ______

6. Consider all five card draws from a standard deck of cards. How many different hands are possible?

   What is the probability that all are hearts? ______________
   What is the probability that at least one is a heart? __________

   What is the probability that exactly three are hearts? __________
More Examples.

ex. A board consists of four members - one represents management (5 available to serve), accounting dept (4 available), sales dept (10), and one from the staff classification (20). How many different boards could be created – in terms of who is currently serving?

ex. How many different menus are possible if there are four salads, three meats, and six desserts and the diner must eat only one of each type?

ex. How many different codes are possible if the code consists of sex, two digit age, and one letter of the alphabet (case sensitive)

ex. How many ways can a Chairman and a Vice Chairman of the Board be selected if they are to come from a group of a 10-member board?

ex. Five men walk into a bar and choose a seat from 12 stools. How many distinct seating arrangements are possible?

ex. Three men and three women come to a party and sit on a long bench. How many seating arrangements are possible? How many if they are to alternate (male-female-male-...)?
ex. An A&M student goes to the library and checks out three out of the five books available. All five books are different (distinct). How many different groups of three are possible?

34/546: Eight horses are entered in a race. In how many ways can the horses finish?

What is the probability that the top three favored horses finish in the order 1, 2, 3?

41/546: How many ways can a 10-question multiple be answered if each question has four different answers?

What is the probability that the first and last questions are correct? (all other questions are wrong)

3) 48/547: A poker hand consists of 5 cards dealt from a deck of 52 cards. How many different poker hands are possible?

How many with four aces? ____________

What is the probability that you have four aces? ____________

4) In how many ways can a hand consisting of 6 spades, 4 hearts, 2 clubs, and 1 diamond be selected from a deck of 52 cards? What is the probability that that happens with a 13-card hand?
5) How many different sequences are possible in 4 rolls of a fair six-sided die? ___________________

What is the probability that the sequence consists of all sixes? ______________

What is the probability that the sequence consists of no sixes? ______________

4/551: Two men and a woman are lined up to have their picture taken. If they are arranged at random, what is the probability that
   a) the woman will be on the left in the picture?  
   b) the woman will be in the middle of the picture?

8/551 Keys for older General Motors cars had six parts, with three patterns each.
   a) How many different key designs are possible for these cars?

   b) If you find an older GM key and you own an older GM car, what is the probability that it will fit your trunk?

14/551 10-questions with 15 possible matches, no repetitions – what is the probability of guessing and getting every answer correct?

20/552 12 girls at random from a freshman class: 200 freshman girls, including 20 from minorities, and the principal would like at least one minority girl to have this honor. If he selects the girls at random, what is the probability that
   a) he will select exactly one minority girl?

   b) he will select no minority girls?

   c) he will select at least one minority girl?
Brief Review of Matrix Operations

Matrices are rectangular array of numbers that describe objects in given sets.

ex. A company produces red and blue bicycles ( Each bike-is of one color). They are either traditional bikes or mountain bikes.

Here are the numbers of bikes produced on Monday;

200 red mountain bikes, 100 blue mountain bikes, 150 red traditional bikes, and 50 blue traditional bikes.

Mountain bikes sell for $150 and traditional bikes sell for $100.

Write a matrix that describes the above situation as well as the amount of revenue for the bikes produced on the given day.

Let M = Mountain Bikes and T = Traditional Bikes

If you are interested in describing the amount of revenue generated by the bikes in terms of their color →

\[
\begin{bmatrix}
M & T \\
\end{bmatrix}
\]

the product of both of these matrix will tell us the amount of money that that is generated by red and blue bicycles.

\[
C \cdot Q
\]

If we wanted to know the value generated by mountain –vs- traditional bikes we would write Q as follows

\[
\begin{bmatrix}
M & T \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
Quantity Matrix Q = \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\end{bmatrix}
\]

\[
C \cdot Q
\]
General Notation:

of matrices

1) rectangular array of numbers with rows and columns
   We normally use capital letters to name the matrices.

\[
A = \begin{bmatrix} 3 \\ \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1/5 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -2 \\ 5 & 7 \end{bmatrix}, \quad D = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 0 \\
-1 & 10 & -2 & -3 & -4 \\
-5 & -6 & -7 & -8 & -9
\end{bmatrix}, \quad E = \begin{bmatrix} 4 \\ 7 \end{bmatrix}
\]

2) Dimension of a matrix: m x n
   We use the number of rows and columns to describe the matrix.

   A is a _________ matrix   B is a _________ matrix

   C: _________   D: _________   E: _________

3) elements of a matrix: a_{ij}
   Look at matrix C: we can label the elements of C as follows:

   Look at matrix E: we can label the elements of E as follows:

   Look at matrix D: find each of the following entries (elements)

       d_{13} = _________   d_{32} = _________   d_{42} = _________   d_{25} = _________

Special Types of Matrices:

Zero Matrices: All entries are zero

2x2 zero matrix   1x5 zero matrix   4x3 zero matrix

Square Matrix: A matrix that has the same number of rows as columns

\[
A = \begin{bmatrix} 5 \\ \end{bmatrix}, \quad B = \begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Identity Matrices: diagonal entries – $a_{11}$, $a_{22}$, $a_{33}$,... are all $= 1$ while all other entries $= 0$

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}, \frac{1}{2} \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}, \frac{1}{2} \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}, \ldots
\]

Addition: add corresponding entries so that you end up with a matrix that resembles the original two in size - this can only occur if the original matrices are identical in size.

$A + B$ is defined if $A$ : $m \times n$ matrix, then $B$ must also be $m \times n$ matrix.

\[
\begin{bmatrix}
3 & -2 \\
-2 & 6
\end{bmatrix} = \begin{bmatrix}
1 & -2 \\
2 & -1 \\
0 & 4
\end{bmatrix} + \begin{bmatrix}
4 & -3 \\
0 & 3
\end{bmatrix} = \begin{bmatrix}
\_ & \_ \\
\_ & \_ \\
\_ & \_
\end{bmatrix}
\]

Subtraction: if treat matrices as real numbers, we can use addition.

Let $-A$ represent the opposite of matrix $A$. Then $B - A = B + (-A)$.

\[
\begin{bmatrix}
2 & -3 \\
4 & 0
\end{bmatrix} - \begin{bmatrix}
3 & -2 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
\_ & \_ \\
\_ & \_ \\
\_ & \_
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 & \_ \\
-2 & \_ \\
0 & \_ \\
\_ & \_ \\
\_ & \_
\end{bmatrix}
\]

There are two types of products of matrices – multiplication by a scalar (nonmatrix – real number) multiplication of two matrices

Scalar Multiplication: easy product - distributive law

a) $4 \begin{bmatrix}
3 \\
-2
\end{bmatrix} = \begin{bmatrix}
\_ \\
\_
\end{bmatrix}$

b) $-2 \begin{bmatrix}
2 & -3 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
\_ & \_ & \_ & \_
\end{bmatrix}$

c) $-2 \begin{bmatrix}
2 & 1 \\
-2 & 1
\end{bmatrix} = \begin{bmatrix}
\_ & \_ \\
\_
\end{bmatrix}$
Some Simple products of Two matrices:

If we multiply matrix A by B (in that order), then the number of columns of A must be the same as the number of rows of A. If A is an m x p matrix, then B must be a p x n matrix.

ex. \[ \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} = ? \]

\[ \begin{bmatrix} -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \end{bmatrix} = ? \]

ex. \[ \begin{bmatrix} 1 & -2 & 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ -3 \end{bmatrix} = \]

ex. \[ \begin{bmatrix} 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \end{bmatrix} = \]

In the two examples above, what do you get if you change the order of the matrices?

ex. \[ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 4 & 1 \end{bmatrix} = \]

General Product of Matrices

ex. \[ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 3 \\ -2 & 2 & 4 \end{bmatrix} = \]

ex. \[ \begin{bmatrix} 1 & -2 & 3 \\ -2 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} = \]
Markov – Chains –
Transition Matrix: has probabilities (transition probabilities)
they describe the probability in moving from one state to another.

Credit Card: Let $C =$ event credit card is used. Previous experience tells a company that

\[
\begin{align*}
P( C \text{ next month } | C \text{ prev. month } ) &= 0.8 & P( C' \text{ next month } | C \text{ prev. month } ) &= 0.2 \\
P( C \text{ next month } | C' \text{ prev. month } ) &= 0.3 & P( C' \text{ next month } | C' \text{ prev. month } ) &= 0.7
\end{align*}
\]

This can be illustrated with a tree diagram as well as with a matrix.

<table>
<thead>
<tr>
<th>Transition matrix uses card</th>
<th>next month card is not used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given card is used</td>
<td>0.8</td>
</tr>
<tr>
<td>Month card is not used</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Initial –probability vector
Prob. woman uses card = 0.9 prob. woman does not use card: _____

\[
\begin{bmatrix}
0.9 \\
0.1
\end{bmatrix}
\]

What are the probabilities for the second month? ____________

Look at example on page 558: Initial Probabilities
\[
\begin{bmatrix}
0.6 & 0.3 & 0.1
\end{bmatrix}
\]

Transition Matrix.
\[
\begin{bmatrix}
0.2 & 0.6 & 0.2 \\
0.1 & 0.5 & 0.4 \\
0.1 & 0.1 & 0.8
\end{bmatrix}
\]

What are the probabilities after one generation?