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Chapter 1

Set Theory

1.1 Sets, Subsets, Intersection, and Union

Definition 1.1.1 (Set - Intuitive)
A set is a collection of distinct objects.

Notation 1.1.1 (Important Sets)
\( \mathbb{N} \) - the natural numbers, \{1, 2, 3, 4, \ldots\}
\( \mathbb{R} \) - the real numbers
\( \mathbb{R}^n \) - Euclidean n-space
\( \mathbb{Q} \) - the rational numbers
\( \mathbb{I} \) - the irrational numbers
\( I \) - the closed interval [0, 1]

Notation 1.1.2
\( \in \) - "Is an element of."
\( \subseteq \) - "Is a subset of."
\( \cup \) - Union
\( \cap \) - Intersection
\( B - A = \{x | x \in A \text{ and } x \in B\} \). (We say that \( B - A \) is the complement of \( A \) in \( B \)).

Theorem 1.1.1
If \( K \subseteq M \subseteq T \) then \( K \subseteq T \)
Theorem 1.1.2
If $A \subset B$ then $B - (B - A) = A.$

Definition 1.1.2 (Arbitrary Unions and Intersections)
Let $\Lambda$ be any set (the index set). For each $\lambda \in \Lambda$ we will define another set $B_\lambda$.

We define $\bigcap B_\lambda = \{x \mid x \in B_\lambda \text{ for each } \lambda \in \Lambda\}.$

We define $\bigcup B_\lambda = \{x \mid \text{There is some } \lambda \in \Lambda \text{ such that } x \in B_\lambda\}.$

1.2 Three Assignment Set/Subset/Equality Theorems

Theorem 1.2.1
Suppose that $A \subset U$ and $B \subset U$.

1. If $A \subset B$ then $(U - B) \subset (U - A)$.
2. If $A \subset B$ then $(A \cup B) = B$
3. If $A \subset B$ then $(A \cap B) = A$

1.3 Cartesian Products, Cardinality, Denumerable

Definition 1.3.1 (Cartesian Product)
Let $A$ and $B$ be nonempty sets. We define $A \times B = \{(a, b) \mid a \in A$ and $b \in B\}$.

Let $A_1, A_2, \ldots, A_n$ be nonempty sets. We define:
$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for each } i\}.$
Definition 1.3.2 (One-to-one Correspondence)

Let A and B be nonempty sets. We say that there is a one-to-one correspondence between A and B if there is a subset C of A × B so that for any a ∈ A there is exactly one pair (a, b) ∈ C and for any b ∈ B there is a pair (a, b) ∈ C.

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Definition 1.3.3 (Same Cardinality)

We say that two sets have the same cardinality if there is a one to one correspondence between them or both sets are empty.

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Definition 1.3.4 (Finite Set)

A set A is finite if there is some n so that there is a one-to-one correspondence between A and \{1, 2, 3, 4, \ldots , n\}.

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Definition 1.3.5 (Denumerable)

A set A is denumerable if A is finite or there is a one-to-one correspondence between A and \mathbb{N}.

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Theorem 1.3.1

The set of positive even numbers is denumerable

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Theorem 1.3.2

The positive rational numbers are denumerable.

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Theorem 1.3.3

A denumerable union of denumerable sets is denumerable.

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Note 1.3.1

We say that the reals have the power of the continuum.

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1.4 Two Assignment Cartesian Product Theorems
Theorem 1.4.1

1. If $A$ and $B$ are finite then $A \times B$ is finite.
2. If $A$ and $B$ are denumerable then $A \times B$ is denumerable.

1.5 Non-denumerable Sets, the Continuum Hypothesis, and Power Sets

Theorem 1.5.1 (Trey’s Favorite)
The interval $(0, 1)$ is non-denumerable.

Definition 1.5.1 (Cardinality of $c$)
Define $c$ to be the cardinality of $(0, 1)$.

Theorem 1.5.2
$$|(0, 1)| = |\mathbb{R}|$$

Definition 1.5.2 (The Power Set)
We define the Power Set of $A$, $\mathcal{P}(A)$, to be the set of all subsets of $A$.

Theorem 1.5.3
If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Theorem 1.5.4
For any $A$, $|\mathcal{P}(A)| > |A|$.
Chapter 2

Topology

2.1 Definition of Topology, Discrete Topology, Trivial Topology

Definition 2.1.1 (Topology)
Let $X$ be a set. We define a topology $\tau \subset \mathcal{P}(X)$, where $\tau$ is the collection of open sets for the topology if the following conditions are met.

1. $\emptyset$ and $X$ are both in $\tau$.
2. $\tau$ is closed under unions.
3. $\tau$ is closed under finite intersections.

Topology 2.1.1 (The Discrete Topology)
Let $\tau = \mathcal{P}(X)$. We call $\tau$ the discrete topology on $X$.

Topology 2.1.2 (The Trivial Topology)
Let $\tau = \{X, \emptyset\}$. This is the trivial topology on $X$.

Topology 2.1.3 (The usual topology on $\mathbb{R}$)
Let $\tau = \{X, \emptyset\} \cup \{(a, b) | a < b\} \cup$ sets of the form $\{(a, b) | a < b\}$. 

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2.2 Eight Topologies

Topology 2.2.1

1. (Kyle Elkins) Two-point Irrational Topology. \(X = P, \tau = \{X, \phi, \{\pi, e\}\}\)

2. (Cody Mitchell) Polynomial Topology. Let \(X = \{P_0, P_1, P_3, \ldots\}\) be an enumeration of polynomials with rational coefficients.
\[
\tau = \{\phi, X, \{P_0\}, \{P_0, P_1\}, \{P_0, P_1, P_2\}, \ldots\}
\]

3. (Don Gray) Twin-Prime, Maybe-It’s-Infinite Topology. \(X = \) the set of prime numbers.
\[
\tau = \{X, \phi\} \cup \{A \mid A = \{P_1, P_2, \ldots\} \text{ where } P_1, P_2, \ldots \text{ are each the sum of two primes.}\}
\]

4. (Paul Dawkins) The Usual Topology Defined Unusually. \(X = \mathbb{R}, \tau = \{(a, b) \mid a, b \text{ are rational}\} \cup \{X, \phi\} \cup \text{ unions of such sets.}\)

5. (Dan McCown) The Bullseye Topology. \(X\) is the real plane. \(\tau = \{X, \phi, A_\lambda, \bigcup A_\lambda\}\) where \(A_\lambda\) is a disc that includes the edge centered at 0 of radius \(\lambda\) with \(\lambda \in \mathbb{N}\)

6. (James Versyp) The Refined Bullseye Topology. \(X\) is the real plane. \(\tau = \{X, \phi, A_\lambda, \bigcup A_\lambda\}\) where \(A_\lambda\) is a disc that includes the edge centered at 0 of radius \(\lambda\) with \(\lambda \in \mathbb{R}^>0\)

7. (Ellen Ellis) The I Don’t Like 0 Topology. \(X = (-1, 0) \cup (0, 1)\).
\[
\tau = \{(-1/n, 0) \cup (0, 1/n) \mid n \in \mathbb{N}\} \cup \{\phi\}
\]

8. (Dustin Butts) The Half-Line Topology. \(X = \mathbb{R}\).
\[
\tau = \{\phi, X\} \cup \{(x, \infty) \mid x \in \mathbb{R}\}
\]
2.3 Closed Set, Set of All Closed Sets, Closure, Interior

Definition 2.3.1 (Closed Set)
Let \( \tau \) be a topology on a set \( X \). We say that a set \( K \) is closed if \( X - K \) is open.

Definition 2.3.2 (3 Properties of \( C \))
Let \((X, \tau)\) be a topology with the set \( C \) (where \( C \) contains all of the closed sets).

1. \( X, \emptyset \in C \).
2. \( C \) is closed under arbitrary intersections.
3. \( C \) is closed under finite unions.

Definition 2.3.3 (Closure and Interior)
Let \((X, \tau)\) define a topology and let \( A \subset X \).

1. Let \( \Lambda \) be the index set for all sets \( K_\lambda \) so that \( K_\lambda \) is closed and \( A \subset K_\lambda \). We define the closure of \( A \) by \( A = \bigcap K_\lambda \).
2. Let \( \Lambda \) be the index set for all sets \( G_\lambda \) so that \( G_\lambda \) is open and \( G_\lambda \subset A \). We define the interior of \( A \) by \( A^\circ = \bigcup G_\lambda \).

Note 2.3.1
Closure - The smallest closed set containing \( A \).

Interior - The largest open set in \( A \).

2.4 Five Properties of Closure, Five Properties of the Interior
Theorem 2.4.1
Let $(X, \tau)$ be a topology with $A, B \subset X$.
If $K$ is any closed set containing $A$ then $\overline{A} \subset K$.

(jv) If $A \subset K_{closed}$ then $\overline{A} \subset K_{closed}$.

Theorem 2.4.2
Let $(X, \tau)$ be a topology with $A, B \subset X$.
If $A \subset B$ then $\overline{A} \subset \overline{B}$.

Theorem 2.4.3
Let $(X, \tau)$ be a topology with $A, B \subset X$.
$\overline{A} = \overline{A}$

Theorem 2.4.4
Let $(X, \tau)$ be a topology with $A, B \subset X$.
$\overline{A \cup B} = \overline{A} \cup \overline{B}$

Theorem 2.4.5
Let $(X, \tau)$ be a topology with $A, B \subset X$.
$\emptyset = \emptyset$

Theorem 2.4.6
Let $(X, \tau)$ be a topology with $A, B \subset X$.
If $G$ is any open set contained in $A$ then $G \subset A^\circ$.
(jv) If $G_{open} \subset A$ then $G_{open} \subset A^\circ$

Theorem 2.4.7
Let $(X, \tau)$ be a topology with $A, B \subset X$.
If $A \subset B$ then $A^\circ \subset B^\circ$.

Theorem 2.4.8
Let $(X, \tau)$ be a topology with $A, B \subset X$.
$(A^\circ)^\circ = A^\circ$

Theorem 2.4.9
Let $(X, \tau)$ be a topology with $A, B \subset X$.
$(A \cap B)^\circ = A^\circ \cap B^\circ$

Theorem 2.4.10
Let $(X, \tau)$ be a topology with $A, B \subset X$.
$X^\circ = X$
2.5 Two Assignment Theorems About Closure and the Interior

Theorem 2.5.1

1. \( A \subset \overline{A} \).
2. \( A^\circ \subset A \).

\( \text{A.04} \)

Theorem 2.5.2

Let \((X, \tau)\) be a topology with \(A, B \subset X\).

1. \( \overline{A \cap B} \) may or may not be equal to \( \overline{A} \cap \overline{B} \)
2. \( \overline{X} = X \)
3. \( (A \cup B)^\circ \) may or may not be equal to \( A^\circ \cup B^\circ \)
4. \( \phi^\circ = \phi \)

\( \text{A.05} \)

2.6 Neighborhood, Neighborhood System, 3 Theorems of \( U_x \)

Definition 2.6.1 (Neighborhood (nhood))

Let \((X, \tau)\) be a topology and \(x \in X\). We say that a set \(U\) is a neighborhood of \(X\) if and only if \(x \in U^\circ\).

\( \text{N.02.03.2003} \)

Note 2.6.1

1. A nhood for \(x\) is not necessarily open.
2. Just because \(x \in U\) does not mean that \(U\) is a nhood of \(x\).

\( \text{N.02.03.2003} \)
Definition 2.6.2 (Neighborhood System)

We define the neighborhood system of $x$ as the set

$$\mathcal{U}_x = \{U | U \text{ is a nhood of } x\}.$$ 

Theorem 2.6.1

Let $(X, \tau)$ be a topology, $x \in X$, and $U \in \mathcal{U}_x$. Then $x \in U$.

Theorem 2.6.2

Let $(X, \tau)$ be a topology, $x \in X$, and $U \in \mathcal{U}_x$. Suppose $U$ and $V$ are both in $\mathcal{U}_x$. Then $U \cap V \in \mathcal{U}_x$.

Theorem 2.6.3

Let $(X, \tau)$ be a topology, $x \in X$, and $U \in \mathcal{U}_x$. If $U \in \mathcal{U}_x$ and $U \subset V$ then $V \in \mathcal{U}_x$.

2.7 Neighborhood Base, 2 Theorems

Definition 2.7.1 (Neighborhood Base)

Fix a point $x \in X$. We say that $\mathcal{B}_x \subset \mathcal{U}_x$ is a nhood base for $x$ if the following 2 conditions hold.

1. Each $V \in \mathcal{B}_x$ is open.

2. For any $U \in \mathcal{U}_x$ there is some $V \in \mathcal{B}_x$ so that $V \subset U$.

Theorem 2.7.1

Let $\mathcal{B}_x$ be a nhood base for $x$ with $V_1$ and $V_2$ in $\mathcal{B}_x$. Then there is some $V_3$ in $\mathcal{B}_x$ so that $V_3 \subset (V_1 \cap V_2)$.

Theorem 2.7.2

$G$ is open if and only if for each $x \in G$ there is some $V \in \mathcal{B}_x$ so that $x \in V \subset G$. 

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2.8 Assignment 6 Neighborhood Base
Theorem

Theorem 2.8.1
If $(X, \tau)$ is a topology where for each $x \in X$ and $B_x = \{\{x\}\}$ is a
hood base, then $\tau$ is the discrete topology.

2.9 Accumulation Points, The Sorgenfrey
Line

Definition 2.9.1 (Accumulation Points)
We say that $x$ is an accumulation point for a set $A$ (or a cluster
point) if and only if for any $V \in B_x$ we have that there is some
point in $V \cap A$ which is different than $x$.

Topology 2.9.1 (The Sorgenfrey Line)
$X = \mathbb{R}$ and $\tau = \{X, \emptyset\} \cup \{[a, b) | a < b\} \cup$ (arbitrary unions in such
sets).