Introduction to Abstract Mathematics

Review for Test 3

1. Determine if the following relations on \( \mathbb{N} \) are reflexive, symmetric, transitive, and/or antisymmetric. Also, determine if the relation is an equivalence relation, a partial ordering, a linear ordering, or none of these.

   (a) \( aRb \) if \( a < b \)
   (b) \( aRb \) if \( a \leq b \)
   (c) \( aRb \) if \( a + b < 10 \)
   (d) \( aRb \) if \( 5 | (a + b) \)
   (e) \( aRb \) if \( b = a^2 \)

2. Define a relation \( R \) on \( \mathbb{N} \) by \( aRb \) if \( a | b \). Prove that \( R \) is a partial ordering.

3. Recall that we define the relation \(<\) on \( \mathbb{Z} \) by \( a < b \) if \( b - a \in \mathbb{N} \). Use this (and P1-P8, Q1-Q9 in section 5.1) to prove that if \( a < b \) then \( -a > -b \).

4. Consider the statement \( S \): If the sum of three integers is less than 30, then at least one of the three is less than 10.

   (a) Rewrite \( S \) using variables and mathematical symbols.
   (b) State the contrapositive of \( S \).
   (c) Prove or disprove \( S \).
   (d) State the converse of \( S \). Prove or disprove.

5. Prove that there is no largest element of \( \mathbb{N} \)

6. Use induction to prove each of the following.

   (a) For all \( n \geq 2 \), \( n^n > n! \)
   (b) \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \)
   (c) \( 1 + 5 + 5^2 + \cdots + 5^n = \frac{5^{n+1} - 1}{4} \)

7. Prove that \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)

8. Prove that \( n^2 + n \) is even for all \( n \in \mathbb{N} \).

9. Prove that 3 divides \( 4^n - 1 \) for all \( n \in \mathbb{N} \).
10. Compute $d = (846, 1314)$. Find $x$ and $y$ s.t. $846x + 1314y = d$.

11. Prove that for any integer $n$, $(n - 1, n) = 1$.

12. Prove that if $n$ is an odd integer, then $(n, n + 2) = 1$.

13. Prove that if $(a, c) = (b, c) = 1$ then $(ab, c) = 1$.

14. Prove or provide a counterexample for the statement: If $(a, c) = (b, c) = 1$ then $(a + b, c) = 1$.

15. Prove that $\sqrt{7}$ is irrational.

16. Prove that there are infinitely many prime numbers.

17. Prove that $\log_{10} 7$ is irrational.

18. Prove that if $x$ is even, then $x^2 \equiv 0$.

19. Prove that there are no integers such that $x^2 = 8y + 3$.

20. For each of the following, write your answer in the form $[a]$ where $0 \leq a < 11$.
   
   (a) $[677]$ (mod 11)
   (b) $[57] + [69]$ (mod 11)
   (c) $[57][69]$ (mod 11)

21. Find a bijection to demonstrate that the even natural numbers form a countably infinite set.

22. Find a bijection to demonstrate that $\mathbb{Z}$ is a countably infinite set.

23. Suppose you have an injective function $f : A \to \mathbb{N}$. Prove that $A$ is countably infinite.