Topology

Review for Test 3

1. Definitions: Be able to define each of the following.
   (a) A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is injective.
   (b) A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is surjective.
   (c) A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is bijective.
   (d) A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is continuous.
   (e) A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is open.
   (f) A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is closed.
   (g) A function $f : (X, \tau) \rightarrow (Y, \tau_1)$ is a homeomorphism.
   (h) $(X, \tau)$ is first countable.
   (i) $d$ is a metric on $X$.
   (j) $(X, \tau_d)$ is a metric topology.
   (k) $K$ is compact.
   (l) Be able to state the Heine-Borel Theorem.
   (m) $A$ is a bounded subset of $\mathbb{R}$.

2. Theorems: Be able to prove each of the following.
   (a) Let $\tau$ be the discrete topology on $X$. Then $f : (X, \tau) \rightarrow (Y, \tau_1)$ is continuous.
   (b) Let $\tau_1$ be the indiscrete topology on $Y$. Then $f : (X, \tau) \rightarrow (Y, \tau_1)$ is continuous.
   (c) Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be a continuous, surjective function. If $(X, \tau)$ is separable then $(Y, \tau_1)$ is as well.
   (d) Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be a continuous, surjective function. If $(X, \tau)$ is connected then $(Y, \tau_1)$ is as well.
   (e) Let $f : (X, \tau) \rightarrow (Y, \tau_1)$ be a continuous, surjective function. If $(X, \tau)$ is compact then $(Y, \tau_1)$ is as well.
   (f) Any metrizable space is first countable.
   (g) Any metrizable space is $T_2$.
   (h) Second countable implies first countable.
   (i) The discrete topology on an infinite set is not compact.
   (j) The indiscrete topology on any set is compact.
   (k) The finite closed topology on any set is compact.
The P/L topology on any set is compact.

Any compact subset of a $T_2$ space is closed.

Any closed subset of a compact space is compact.

Compactness is not hereditary.

Any compact $T_2$ space is $T_3$.

Let $f : (X, \tau) \to (Y, \tau_1)$ be a continuous function. If $(X, \tau)$ and $(Y, \tau_1)$ are both compact and $T_2$, then $f$ is a closed mapping.

3. Problems

(a) Let $f : (X, \tau) \to (Y, \tau_1)$ be a constant function. Then $f$ is continuous.

(b) Let $f : (X, \tau) \to (X, \tau)$ be the identity function. Then $f$ is continuous.

(c) Let $f : (X, \tau) \to (Y, \tau_1)$ be a continuous injective function with $(Y, \tau_1) T_2$. Then $(X, \tau)$ is $T_2$.

(d) Provide counterexamples to show that $T_i$, $i = 0, 1, 2, 3, 4$ is not preserved by continuous surjections.

(e) For each of the following metrics on $\mathbb{R}^2$, sketch $B_2((0, 0))$.

i. $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

ii. $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$

iii. $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

iv. $d((x_1, y_1), (x_2, y_2)) = \begin{cases} 0 & \text{if } x_1 = x_2, y_1 = y_2 \\ 1 & \text{o.w.} \end{cases}$

(f) Provide a counterexample to show that just because $(X, \tau)$ is metrizable, and $f : (X, \tau) \to (Y, \tau_1)$ is continuous and surjective, that does not guarantee that $(Y, \tau_1)$ will be metrizable.

(g) Know which of our spaces are metrizable and which are not (see Review for Test 2 for a list of spaces).

(h) Know which of our spaces are compact and which are not (see Review for Test 2 for a list of spaces).

(i) Determine whether the following subsets of $\mathbb{R}$ are compact or not. Justify your answers.

i. $[3, 45]$

ii. $(3, 45)$

iii. $[5, \infty)$

iv. $\mathbb{N}$

v. $\{1/2, 1/3, 1/4, 1/5, \ldots\}$

(j) Find examples of functions that are...

i. open but not closed.

ii. closed but not open.

iii. open but not continuous.
iv. closed but not continuous.
v. continuous but not open.
vi. continuous but not closed.

Also, be able to determine the openness, closedness and continuity of a function that you are given. For example: $f : \mathbb{R} \to \text{Sorgenfrey}$, defined by $f(x) = 2x$. Is this open? closed? continuous?